Applications of Fourier Transforms to Generalized Functions

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This book is dedicated to Sir James Lighthill, FRS who gave the author tremendous inspiration in applied mathematics during his days at Imperial College London.

"We make a living by what we get
but we make a life by what we give"

Sir Winston Churchill

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Contents

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Preface

The generalized function is one of the important branches of mathematics that has enormous application in practical fields. Especially, its applications to the theory of distribution and signal processing are very much noteworthy. The method of generating solutions is the Fourier transform, which has great applications to the generalized functions. These two branches of mathematics are very important for solving practical problems. While I was at Imperial College London (1966–1969), I attended many lectures delivered on fluid mechanics topics by Sir James Lighthill, FRS. At that time I was unable to understand many of the mathematical ideas in connection with the generalized function and why we need this abstract mathematics in the applied field. I tried to follow Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*, published by Cambridge University Press, 1964. His book is very compact (only 79 pages) and extremely stimulating, but he has written it so elegantly that unless one has good mathematical background, the book is very hard to follow. I understand that a non-expert reader will find the book very hard to follow because of its compactness and too many cross-references. Mathematical details are very minimal and he sequentially explains from one step to another skipping many intermediate steps by the cross references. Lighthill followed the ideas originally described by Professor George Temple's *Generalized Functions, Proc. Roy. Soc. A,* **228**, *175–190, (1955)*. Lighthill kept the theory part as described by Temple. In Dalhousie University I used to give a course on *Mathematical Methods and their Applications* to the undergraduate and graduate students for several years. I used Fourier transforms and generalized functions in that course. To make it understandable to the student I had to take recourse to some engineering textbooks where the applications are found in this subject. I followed some engineering application of generalized functions and its solution technique using the Fourier transform method.

This book grew partly out of my course given to the undergraduate and graduate students at Dalhousie University, Halifax, Nova Scotia, Canada; and partly from reading the books by Temple and Lighthill. This book explains clearly the intermediate steps not found in any other book. The book leans heavily towards Lighthill's book, but I have bridged the gap of mathematical deductions by clearly manifesting every important step with illustrations and mathematical tables. I think a layman can also follow my book without much difficulty. I must admit that this book is written in such a way as if I have revisited Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*. This book, hopefully, will be useful to the non-expert and also the experts alike. With this intention, the book is prepared in my own way collecting some additional material from some other textbooks including Professor D.S. Jones' book on *Generalized Functions*, published by McGraw-Hill Book Company, New York, 1966. I have borrowed some ideas from Professor Jones' book. Specially, I borrowed some important practical unsolved examples that I solved myself for the benefit of the reader. It is my hope that the reader will gain some insight about this important but esoteric mathematical subject.

The first chapter of the book deals with the introductory concept of Fourier series, Fourier integrals, Fourier transforms and the generalized function. The theoretical development of the Fourier transform is described and the first generalized function is defined with some illustrations. Some important examples are manifested in this chapter. Some interesting exercises are included at the end of the chapter.

Chapter 2 deals with the formal definition of the generalized function. A clear-cut definition of a good function and a fairly good function as illustrated by Lighthill is demonstrated in this chapter. The difference between an ordinary function and a generalized function is given with some examples. Even and odd generalized functions are clearly defined. The chapter ends with some useful exercises.

Chapter 3 consists of Fourier transforms of particular generalized functions. This chapter deals with the integral power of an algebraic function, non-integral powers, the Fourier transforms of $x^n \ell n|x|$, $x^{-m}\ell n\,|x|$, $x^{-m}\ell n|x|sgn(x)$ together with the summary of results of Fourier transforms. The chapter concludes with some exercises.

Asymptotic estimation of Fourier transforms are discussed in details in Chapter 4. First we have defined the Riemann-Lebesgue lemma which is important to obtain the asymptotic value of a generalized function. The asymptotic expression of the Fourier transform of a function with a finite number of singularities is discussed. We demonstrated solutions of some generalized functions using asymptotic expressions. Fourier transforms play a major role. Some important numerical solutions of some integrals are listed in Table 4.1. Whereas Table 4.2 contains a short list of Fourier transforms of 18 important generalized functions at a glance. The chapter ends with some important exercises.

Chapter 5 contains the Fourier series as a series of generalized functions. We demonstrated how to evaluate the coefficients of a trigonometric series. Some practical examples such as Poisson's summation formula and the asymptotic behaviour of the coefficients in a Fourier series are illustrated. This chapter concludes with some exercises.

We conclude the book (Chapter 6) with an important topic concerning the fast Fourier transform. It is a numerical procedure which is fast, accurate and efficient to determine the Fourier coefficients that are the Fourier transforms using an algorithm developed by Cooley and Tukey in 1965. Some preliminary studies of the Fourier transform with ample examples are also demonstrated in this chapter by using analytical and graphical methods. We have not reiterated the algorithm of Cooley and Tukey, rather we have given a numerical view of how it works, citing a practical example in the study of wave energy spectrum density as illustrated elegantly by Chakrabarti (1987). A handful of exercises are included and some references are cited at the end of the chapter.

The book concludes with three appendices. Appendix A deals with Fourier transforms of some important generalized functions. Appendix B is concerned with some important properties of Dirac delta $\delta(x)$ functions and Appendix C contains a comprehensive list of some important references concerning with the generalized functions and the application of the fast Fourier transform for further reading. A subject index is also included at the end of the book.

While it has been a joy to write such a comprehensive book for a long period of time, the fruits of this labour will hopefully be in learning of the enjoyment and benefits realized by the reader. Thus the author welcomes any suggestions for the improvement of the text.

Matiur Rahman, 2011
Halifax, Canada

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Dr Seyed Hossein Mousavizadegan and Dr Adhi Susilo deserve my special thanks for their assistance in drafting and designing some figures for this book. I also appreciate the assistance from Ms Rhonda Sutherland, Miss Rekha Arora of Computer Science, and Mrs Karen Conrod for helping me in drafting some figures contained in this manuscript.

I am extremely grateful to Dr Michael Shepherd, Dean of Faculty of Computer Science at Dalhousie University for allowing me to use some facilities to complete this manuscript. Thanks are also extended to Dr Denis Riordan, Professor and Associate Dean of Faculty of Computer Science for his encouragement and constructive and favorable comments about the manuscript.

This book is primarily derived from Lighthill's book on *Introduction to Fourier analysis and generalised functions* published by Cambridge University Press in 1958 and subsequently reprinted in 1964. Thus, Cambridge University Press deserves my appreciations for the use of ideas and concepts which help me develop the present manuscript. WIT Press is gratefully acknowledged for its superb job in producing such a beautiful book.

1 Introduction

1.1 Preliminary remarks

Generalized functions are an important area of mathematical sciences. They have a wide range of applications. I taught at Dalhousie University several applied mathematics courses including mathematical methods which contain Laplace transforms, Fourier series and integrals, and Fourier transforms. These subject matters were developed during the courses at the undergraduate and graduate levels. The generalized function was used in connection with the problems of distribution theory and signal processing. It seems that a satisfactory account must make use of functions, such as the delta function of Dirac, which are outside the usual scope of functional theory. The *Theorie des distributions* by Schwartz (1950–1951) is one of the books that has developed a detailed theory of generalized functions. Professor Temple (1953, 1955) has given a version of the theory of generalized functions which appears to be more readily intelligible to senior undergraduates. Temple's book curtails the labour of understanding Fourier transforms. His book makes available a technique for the asymptotic estimation of Fourier transforms which seems superior to previous techniques. This is an approach in which the theory of Fourier series appears as a special case, the Fourier transform of a periodic function being a *row of delta functions*. Lighthill (1964) discussed this matter elaborately.

The main purpose of this book is to explain some of the very abstract material presented by Lighthill in his book. I think it will be a very useful addition to the literature and will help the graduate students of our future generation to have a clear-cut exposition of the subject of generalized functions and Fourier transforms. This book not only covers the principal results concerning Fourier transforms and Fourier series, but also serves as an introduction to the theory of generalized functions that are used in Fourier analysis. It contains simple properties without rigorous mathematical proofs.

1.2 Introductory remarks on Fourier series

A Fourier series is a representation of a periodic function $f(x)$ of period 2ℓ, say, which means that $f(x) = f(x+2\ell) = \cdots = f(x+2n\ell)$, for $n = 1, 2, 3, \ldots$, as a linear combination of all *cosine* and *sine* functions which have the same period, say, as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(n\pi x/\ell)\,dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(n\pi x/\ell)\,dx, \quad n = 1, 2, \dots. \tag{1.1}$$

The fundamental period in the trigonometric expression on the right-hand side is 2ℓ, because for $n=1$, $\cos\left(\frac{\pi x}{\ell}\right) = \cos\left(\frac{\pi x}{\ell} + 2\pi\right) = \cos\left(\frac{\pi}{\ell}(x+2\ell)\right)$, and similarly for $\sin\left(\frac{\pi x}{\ell}\right)$. The constant a_0 has no fundamental period but it is a periodic function with any natural number including 2ℓ. Fourier series in this sense are used for analysing oscillations periodic in time, or waveforms periodic in space, and also for representing functions of plane or cylindrical coordinates, when x in eqn (1.1) becomes the polar angle θ, and the period 2ℓ becomes 2π. It is interesting to note here that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{1.2}$$

such that $f(\theta) = f(\theta + 2\pi)$.

The series (1.1) can be written, more compactly, in the complex form as follows:

$$\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{in\pi x/\ell} + e^{-in\pi x/\ell}}{2} + b_n \frac{e^{in\pi x/\ell} - e^{-in\pi x/\ell}}{2i} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-in\pi x/\ell} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} + \sum_{n=-\infty}^{-1} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} \\
&= \frac{a_0}{2} + \sum_{n=-\infty}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} \\
&= C_0 + \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell} \\
&= \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},
\end{aligned}$$

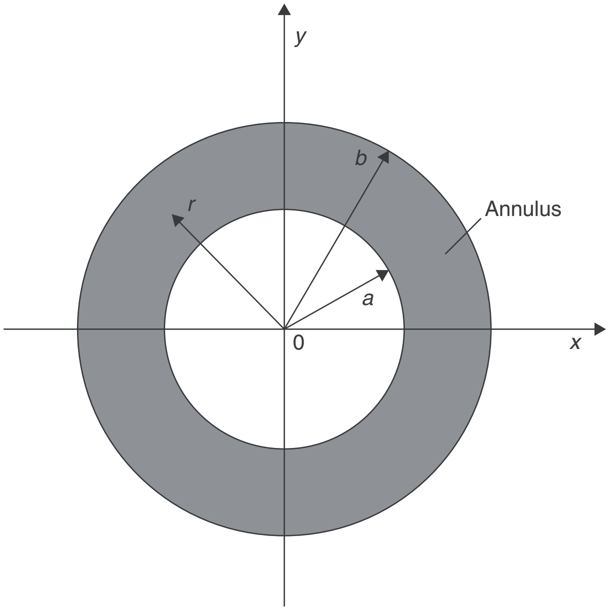

Figure 1.1: An annular region between two concentric circles of radii $r = a$ and b $(a \le r \le b)$.

where $C_0 = \frac{a_0}{2}$, $C_n = (a_n - ib_n)/2$ and $C_{-n} = (a_n + ib_n)/2 = C_n^*$, in which C_n^* is a complex conjugate. It can be easily verified that $C_n + C_n^* = a_n$ and $C_n - C_n^* = -ib_n$. With this information the trigonometric series (periodic function) can be represented as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},$$
$$C_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\, dx. \tag{1.3}$$

One great advantage of expressing a function in terms of cosine and sine functions, or even more in terms of exponentials, is the simple behaviour of these functions under the various operations of analysis, notably differentiations and integrations. The Fourier series expression (1.3) may be used for some practical problems. For example, solutions to Laplace's equation in plane polar coordinates r; θ, which are periodic in θ with period 2π, thus, representing solutions which are one-valued in an annulus with the centre as the origin, may be written as a boundary value problem as illustrated in Figure 1.1.

The partial differential equation is

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0, \quad 0 \le \theta \le 2\pi, \;\; a \le r \le b. \tag{1.4}$$

The boundary conditions are

$$r = a: \quad f(a,\theta) = g(\theta), \tag{1.5}$$

$$r = b: \quad \frac{\partial f}{\partial r}(b,\theta) = h(\theta). \tag{1.6}$$

The periodic solution to eqn (1.4) may be written as

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} C_n(r)e^{in\theta}. \tag{1.7}$$

If we substitute this in eqn (1.4), assuming that we can differentiate term by term, we obtain

$$\sum_{n=-\infty}^{\infty} \left[\frac{d^2C_n}{dr^2} + \frac{1}{r}\frac{dC_n}{dr} - \frac{n^2}{r^2}C_n\right]e^{in\theta} = 0. \tag{1.8}$$

Now if we assume that expressions of functions by such trigonometric series are unique, then a series which vanishes identically must have vanishing coefficients, while an ordinary differential equation is obtained for C_n, $\frac{d^2C_n}{dr^2} + \frac{1}{r}\frac{dC_n}{dr} - \frac{n^2}{r^2}C_n = 0$, whose general solution is

$$C_n(r) = A_n r^n + B_n r^{-n}. \tag{1.9}$$

Therefore, we have

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} [A_n r^n + B_n r^{-n}]e^{in\theta}. \tag{1.10}$$

If the boundary conditions (1.5) and (1.6) are applied on the circles $r = a$ and b, respectively, to determine the unknown constants A_n and B_n, we must express $g(\theta)$ and $h(\theta)$ in Fourier series so that

$$g(\theta) = \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \quad h(\theta) = \sum_{n=-\infty}^{\infty} h_n e^{in\theta}. \tag{1.11}$$

Then we have two algebraic equations

$$\begin{aligned} A_n a^n + B_n a^{-n} &= g_n, \\ A_n b^n - B_n b^{-n-1} &= \frac{h_n}{n}, \end{aligned} \tag{1.12}$$

the solutions to which are given by

$$\begin{aligned} A_n &= [b^{-n-1}g_n + a^{-n-1}(h_n/n)]/D, \\ B_n &= [b^{n-1}g_n - a^n(h_n/n)]/D, \end{aligned} \tag{1.13}$$

where $D = a^n b^{-n-1} + a^{-n-1}b^{n-1}$.

This example is so simple that it could be treated in many different ways, but clearly, the procedure is directly applicable in most complicated problems, provided always that the argument of the Fourier series (here θ) varies independently of the other variables from the boundary conditions. This example makes it clear that a satisfactory Fourier series theory will be one in which term-by-term differentiation and unique determination of coefficients for a given function are both possible. These two requirements had never been simultaneously satisfied by any of the Fourier series theories until the *generalized function* approach was developed.

1.3 Half-range Fourier series

Some concepts about the half-range Fourier series and quarter-range Fourier series are described by the following example. If a partial differential equation is to be solved in a region part of whose boundary consists of the lines (or planes) $x=0$ and ℓ, then the argument is usually presented as follows. The only cosines or sines which satisfy the boundary condition $f=0$ both at $x=0$ and ℓ are $\sin(n\pi x/\ell)$ $(n=1,2,3,\ldots)$, so that the half-range Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell) \quad (0 < x < \ell) \tag{1.14}$$

is applicable when these are the boundary conditions. Alternatively, if $\frac{\partial f}{\partial x}=0$ at $x=0$ and ℓ, then the half-range Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/\ell) \quad (0 < x < \ell) \tag{1.15}$$

is applicable. Note that in the complex form (1.3), in these cases C_n is purely imaginary and real, respectively.

Again, if $f=0$ at $x=0$ and $\frac{\partial f}{\partial x}=0$ at $x=\ell$, the quarter-range Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2\ell} \quad (0 < x < \ell), \tag{1.16}$$

containing an even selection of the terms in a Fourier series of larger period 4ℓ, is applicable. These predictions can be easily verified by considering the problem of oscillations of a one-dimensional string of length ℓ fixed at both ends by using appropriate boundary conditions.

1.3.1 Verification of conjecture (1.14)

To verify the solution (1.14), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} &= c^2 \frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell) \ (t > 0), \\ x = 0: \quad f &= 0, \\ x = \ell: \quad f &= 0. \end{aligned} \tag{1.17}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.18}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell) \quad (0 < x < \ell).$$

This is a half-range Fourier sine series and b_n is the Fourier coefficients as given in eqn (1.14). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\sin n\pi x/\ell\}$, for $n = 1, 2, 3, \ldots$, defined in the interval $0 \leq x \leq \ell$. The formula for b_n is given by $b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(n\pi x/\ell)\,dx$, $n = 1, 2, 3, \ldots$.

1.3.2 Verification of conjecture (1.15)

To verify the solution (1.15), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} &= c^2 \frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell) \ (t > 0), \\ x = 0: \quad \frac{\partial f}{\partial x} &= 0, \\ x = \ell: \quad \frac{\partial f}{\partial x} &= 0. \end{aligned} \tag{1.19}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.20}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/\ell) \quad (0 < x < \ell).$$

This is a half-range Fourier cosine series and a_n is the Fourier coefficients as given in eqn (1.15). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\cos(n\pi x/\ell)\}$, for $n = 0, 1, 2, 3, \ldots$, defined in the interval $0 \le x \le \ell$. The formula for a_n is given by $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(n\pi x/\ell)\,dx, n = 0, 1, 2, 3, \ldots$.

1.3.3 Verification of conjecture (1.16)

To verify the solution (1.16), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} &= c^2 \frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell) \ (t > 0), \\ x = 0: \quad f &= 0, \\ x = \ell: \quad \frac{\partial f}{\partial x} &= 0. \end{aligned} \tag{1.21}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.22}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \sum_{n=1}^{\infty} b_n (\sin(2n-1)\pi x/2\ell) \quad (0 < x < \ell).$$

This is a quarter-range Fourier series and b_n is the Fourier coefficients as given in eqn (1.16). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\sin((2n-1)\pi x/2\ell)\}$, for $n = 1, 2, 3, \ldots$, defined in the interval $0 \le x \le \ell$. The formula for b_n is given by $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin((2n-1)\pi x/2\ell)\,dx,\ n = 1, 2, 3, \ldots$.

Remark

These series can be approached in a slightly more useful manner as described below. To satisfy the boundary conditions $f(0) = 0$ and $f(\ell) = 0$, an *odd periodic function* $f(x)$ of period 2ℓ is introduced. Its Fourier series (1.1) then contains only odd terms and reduces to eqn (1.14).

Similarly, the boundary conditions $\frac{\partial f}{\partial x} = 0$ at $x = 0$ and ℓ can be satisfied by using an *even periodic function*. Finally, the boundary conditions $f = 0$ at $x = 0$ and $\frac{\partial f}{\partial x} = 0$ at $x = \ell$ can be satisfied by using an odd periodic function, of period 4ℓ, which is also an even function of $(x - \ell)$; note that only those sine terms of period 4ℓ which satisfy the latter condition appear in eqn (1.16). In each case, naturally, it is the value of the periodic function in the range $0 < x < \ell$ that represents the solution to the problem.

1.4 Construction of an odd periodic function

A simple example is now given to show the advantage of replacing boundary conditions wherever possible by the condition of periodicity and partly, even in a case where Fourier series are not used. If a string is stretched between two fixed points $x = 0$ and ℓ and plucked (i.e. released from rest in a distorted shape), we can imagine an infinite stretched string where transverse displacement $y = f(x)$ is periodic with period 2ℓ and odd and agrees with the given shape for $0 < x < \ell$ (see Figure 1.2).

If this infinite string were released from rest in this position, the displacement y would remain odd and periodic, and so continue to satisfy the boundary condition $y = 0$ at $x = 0$ and ℓ. Hence the simple solution to the initial-value problem for the infinite string, namely

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2}, \quad -\infty < x < \infty, \\ t = 0: \quad y(x, 0) &= f(x), \\ \frac{\partial y}{\partial t}(x, 0) &= 0, \end{aligned} \tag{1.23}$$

would be D'Alembert's wave given by

$$y = \frac{1}{2}\{f(x - ct) + f(x + ct)\}. \tag{1.24}$$

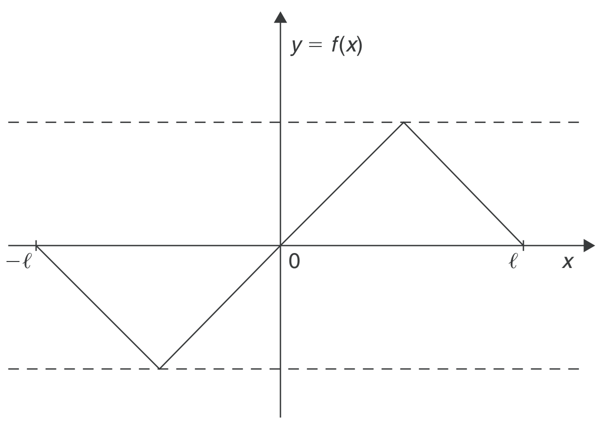

Figure 1.2: Illustrating the construction of an odd periodic function $f(x)$ taking given values in $0 < x < \ell$.

For detailed calculations leading to this important solution, the reader is referred to the delightful book *Mathematical Methods with Applications* by Rahman (2001). The result (1.24) can be used to give the solution, avoiding the need to consider multiple reflections from the ends. Similar advantages will accrue if other problems of this kind are treated in this way. Accordingly, it is best to consider every Fourier series as the Fourier series of a periodic function, even in cases where one is primarily interested only in values over a half or quarter period. This is especially advisable in constructing any general theory of Fourier series, since the sum of such a series, if it exists, is certainly periodic.

We must conclude this section by listing the principal aims of a theory of Fourier series. First, one must obtain the condition under which a trigonometrical series (1.1) or (1.2) converges. For example, a sufficient condition for absolute uniform convergence is that $C_n = O(|n|^{-1-\varepsilon})$ as $|n| \to \infty$ for some $\varepsilon > 0$. Note that Lighthill (1964) defines the order symbols O and o in an extremely elegant way.

Remark

For ready reference, we give the definitions of these two symbols as follows: $f = O(g)$ means that $|f| < A|g|$ for some positive A as the limit is approached. On the other hand, $f = o(g)$ means that $|f| < \varepsilon|g|$ (sufficiently near to the limit) for any positive ε (however small). A necessary and sufficient condition for convergence in the series of generalized function theory is that $C_n = O(|n|^N)$ as $|n| \to \infty$ for some N. Finally, one may ask the question concerning the term-by-term differentiability and uniqueness. These are the properties which are most notably lacking in the "convergence" and "summability" theories, respectively; but both results are almost trivial in the generalized function theory. For further study, the reader is referred to Hardy & Rogosinski (1950).

1.5 Theoretical development of Fourier transforms

The Fourier integral may be regarded as the formal limit of the Fourier series as the period tends to infinity. Thus if $f(x)$ is any function of x in the whole range $(-\infty < x < \infty)$, one can form a periodic function $f_\ell(x)$ of the period 2ℓ which agrees with $f(x)$ in the range $(-\ell, \ell)$. The Fourier series of $f_\ell(x)$, with the expression (1.3) for C_n, can be written in the form

$$\begin{aligned}
f_\ell(x) &= \sum_{n=-\infty}^{\infty} C_n e^{in\pi y/\ell} \\
&= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2\ell}\right) \int_{-\ell}^{\ell} f_\ell(z) e^{-in\pi z/\ell}\, dz \right) e^{in\pi x/\ell} \\
&= \sum_{n=-\infty}^{\infty} \left\{ \int_{-\ell}^{\ell} f_\ell(z) e^{-in\pi z/\ell}\, dz \right\} (e^{in\pi x/\ell}) \left(\frac{1}{2\ell}\right).
\end{aligned}$$

Let us assume that $\frac{n}{2\ell} = y_n$ such that $\frac{n+1}{2\ell} = y_{n+1}$ and hence $\Delta y = y_{n+1} - y_n = \frac{1}{2\ell}$, then we have

$$f_\ell(x) = \sum_{n=-\infty}^{\infty} \left\{ \int_{-\ell}^{\ell} f_\ell(z) e^{-i2\pi yz}\, dz \right\} (e^{i2\pi xy})\, \Delta y.$$

Let ℓ tend to infinity, such that the above equation becomes

$$\begin{aligned} f(x) &= \int_{y=-\infty}^{\infty} \left\{ \int_{z=-\infty}^{\infty} f(z) e^{-i2\pi yz}\, dz \right\} (e^{i2\pi xy})\, dy \\ &= \int_{y=-\infty}^{\infty} g(y)(e^{i2\pi xy})\, dy, \end{aligned}$$

where

$$g(y) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xy}\, dx.$$

This is called the Fourier transform pair.

Note that ℓ tends to infinity implies $\Delta y = \frac{1}{2\ell}$ tends to dy. And also $f_\ell(x) = f(x)$ and $g_\ell(y) = g(y)$. Thus summarizing the situation, we have

$$\begin{aligned} \mathcal{F}(f(x)) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx = g(y), \\ f(x) = \mathcal{F}^{-1}(g(y)) &= \int_{-\infty}^{\infty} g(y) e^{2\pi ixy}\, dy. \end{aligned}$$

Under these circumstances the function $g(y)$ is often called the Fourier transform of $f(x)$. We have used here the symbol $\mathcal{F}$ to denote the Fourier transform. The above equation may be regarded as stating that $f(y)$ is the Fourier transform of $g(-x)$ which can be verified as follows. We know that $f(x) = \int_{-\infty}^{\infty} g(y) e^{2\pi ixy}\, dy$ is usually known as the Fourier inverse. Changing x to y and then y to $(-x)$ in the integral, we obtain

$$\begin{aligned} f(y) &= \int_{\infty}^{-\infty} g(-x) e^{-2\pi ixy}(-dx) \\ &= \int_{-\infty}^{\infty} g(-x) e^{-2\pi ixy}\, dx \\ &= \mathcal{F}(g(-x)). \end{aligned}$$

Thus mathematically we can write $\mathcal{F}(g(-x)) = f(y)$.

Remark

For further information, the reader is referred to the classical book by Titchmarsh (1937). The reader should be warned, however, that no general agreement has been reached on where the 2π's in the definition of Fourier transforms should be placed. This point very simply can be addressed as follows. We have the Fourier transform pair as $g(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$ and $f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy$. Let us substitute $2\pi y = k$ such that $dy = \frac{1}{2\pi}\,dk$, and using this information into the Fourier transform pair, one can easily obtain the familiar pair as $g(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)e^{ikx}\,dk$. Thus our familiar Fourier transform pair can be recovered. It is worth mentioning that in the space domain k is defined as the wavenumber. However, in the time domain k is the radial frequency leading to the pair $g(\sigma) = \int_{-\infty}^{\infty} f(t)e^{-i\sigma t}\,dt$ and $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\sigma)e^{i\sigma t}\,d\sigma$.

Now coming back to the original development of the Fourier transform pair we have found that a vast literature is available for the determination of conditions of $f(x)$ sufficient for equations $g(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$ and $f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy$ to be true with a given interpretation of the integrals. Even for a fixed interpretation, many alternative sets of sufficient conditions are necessary if one wishes to apply the equations widely, because relaxing one condition to allow one some desired functions requires usually the strengthening of some other condition, which in turn excludes some other functions. All these difficulties disappear when the generalized function $f(x)$ has a Fourier transform $g(y)$ which is also a generalized function, and the Fourier transform of $g(-x)$ is

$$f(y) = \int_{-\infty}^{\infty} g(-x)e^{-2\pi ixy}\,dx.$$

In the latter theory it is appropriate to proceed in an order different from that adopted in this chapter, and to treat the properties of Fourier series as a special case of the Fourier transform.

1.6 Half-range Fourier sine and cosine integrals

The Fourier integral is used to analyse non-periodic functions of x in the range $(-\infty, \infty)$ as linear combinations of exponential functions. This type of analysis is useful for much the same reasons as the Fourier series. For example, it is effective in treating linear partial differential equations with the coefficient independent of x, subject to boundary conditions given on the boundaries when x varies from $-\infty$ to ∞ independently of other variables. To apply these boundary conditions, it is necessary to be able to express any functions occurring in them as Fourier integrals. But a difficulty arises if the function is a constant which has no Fourier transform in ordinary function theories. This difficulty disappears in the theory of generalized functions, in which, for example, the Fourier transform of 1 is the delta function of Dirac. This point is illustrated below. Let us consider, for example, the Fourier transform of 1:

$$\begin{aligned}\mathcal{F}\{1\} &= \int_{-\infty}^{\infty} (1)e^{-2\pi ixy}\,dx \\ &= \lim_{x\to\infty} \frac{e^{2\pi ixy} - e^{-2\pi ixy}}{2\pi iy} \\ &= \lim_{x\to\infty} \frac{\sin(2\pi xy)}{\pi y} \\ &= 2\pi\delta(2\pi y) \\ &= \delta(y).\end{aligned}$$

We have arrived at this result by using the definition of $\delta(y) = \lim_{x\to\infty} \frac{\sin(xy)}{\pi y}$. Now we turn our attention to the half-range Fourier integrals. Half-range Fourier integrals are used along much the same lines as the Fourier series. Thus, if $f(x)$ is to be determined in the range $(0,\infty)$ subject to a condition $f(0) = 0$, one may select an odd function $f(x)$ in the full range $(-\infty,\infty)$, which coincides with $f(x)$ in $(0,\infty)$. Hence using the Fourier transform pair if $g(y)$ is the Fourier transform of $f(x)$, we have

$$\begin{aligned}g(y) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx, \\ f(x) &= \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy \qquad (1.25)\end{aligned}$$

from which

$$\begin{aligned}g(y) &= \int_{-\infty}^{\infty} f(x)[\cos(2\pi xy) - i\sin(2\pi xy)]\,dx \\ &= -2i\int_{0}^{\infty} f(x)\sin(2\pi xy)\,dx \qquad (1.26)\end{aligned}$$

because the first integral on the right-hand side is zero. Here $g(y)$ is an odd function if $f(x)$ is odd. And hence using the similar deduction, we obtain

$$f(x) = 2i\int_{0}^{\infty} g(y)\sin(2\pi xy)\,dy. \qquad (1.27)$$

Similarly, if $f(x)$ is to be determined in $(0,\infty)$ subject to a condition $\frac{\partial f}{\partial x} = 0$ when $x = 0$, one may seek an even function $f(x)$ in the full-range $(-\infty,\infty)$ which coincides with $f(x)$ in $(0,\infty)$. Its Fourier transform may be written as

$$g(y) = 2\int_{0}^{\infty} f(x)\cos(2\pi xy)\,dx, \qquad (1.28)$$

which is also an even function, and the expression for $f(x)$ in terms of $g(y)$ becomes

$$f(x) = 2\int_0^{\infty} g(y)\cos(2\pi xy)\,dy. \tag{1.29}$$

The integrals in eqns (1.28) and (1.29) are called the Fourier cosine integrals.

As with Fourier series, no special theory is needed for Fourier sine integrals and Fourier cosine integrals. They should be regarded simply as what is obtained by taking Fourier transforms of odd and even functions, respectively. In many cases, especially when it is not possible to evaluate a Fourier transform explicitly in terms of available tabulated transform functions, it is useful to have a technique for evaluating the asymptotic behaviour of Fourier transform $g(y)$ as $|y| \to \infty$, in terms of the behaviour of $f(x)$ near its singularities. It is difficult to find a comprehensive account of this technique in the literature, and since the theory becomes particularly simple when generalized functions are used, a substantial portion of Lighthill's (1964) book has been devoted to expounding it. This theory can also be applied without changing the problem of determining the asymptotic behaviour as $|n| \to \infty$ of the coefficient C_n in Fourier series for a given function.

1.7 Introduction to the first generalized functions

The first *generalized function* to be introduced here is Dirac's "delta function" $\delta(x)$, which has the property that the area under this curve is always 1, that is, $\int_{-\infty}^{\infty}\delta(x)\,dx = 1$. Also for any continuous function $F(x)$ it has the property that

$$\int_{-\infty}^{\infty}\delta(x)F(x)\,dx = F(0)\int_{-\infty}^{\infty}\delta(x)\,dx = F(0). \tag{1.30}$$

No function in the ordinary sense has the property (1.30), but one can imagine a sequence of functions (see Figure 1.3) which have progressively taller and thinner peak at $x = 0$, with the area under the curve remaining equal to 1, while the value of the function tends to 0 at every point, except at $x = 0$ where it tends to infinity. In the limit this sequence would have the property (1.30). It can be easily verified that the area covered by each of the sequence of this function is unity. The following calculation will verify this conjecture.

Let us consider that $\delta(x) = \lim_{n\to\infty}\delta_n(x) = e^{-nx^2}\left(\frac{n}{\pi}\right)^{1/2}$. The area A covered by the sequence of functions $\delta_n(x)$ is calculated as follows:

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} e^{-nx^2}\left(\frac{n}{\pi}\right)^{1/2} dx \\
&= 2\left(\frac{n}{\pi}\right)^{1/2}\int_0^{\infty} e^{-nx^2}\,dx \\
&= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\eta^2}\,d\eta \\
&= erf(\infty) = 1,
\end{aligned}$$

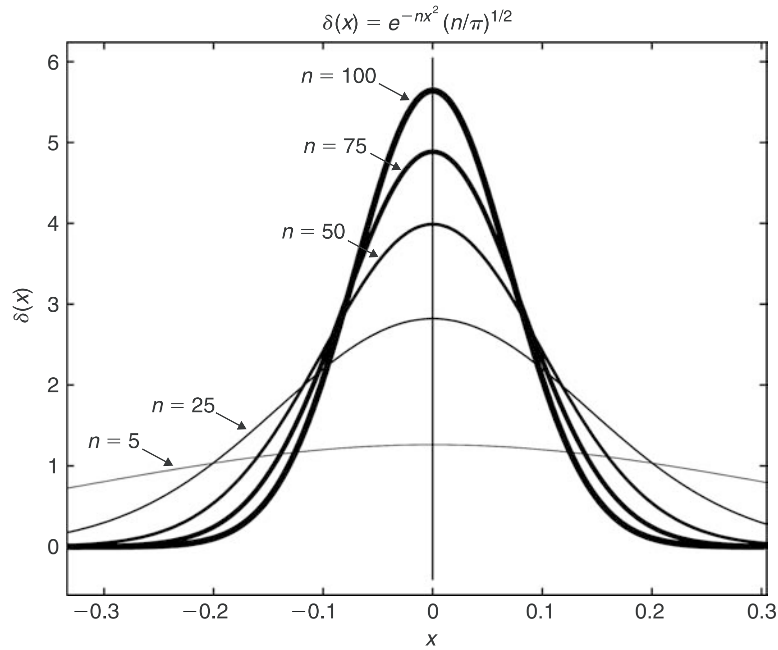

Figure 1.3: Functions in the sequence $\delta_n(x) = e^{-nx^2}\left(\frac{n}{\pi}\right)^{1/2}$ used to define $\delta(x)$; the graphs are depicted for the number n (for $n = 5$, 25, 50, 75, 100) for the nth function. (From Lighthill, 1964.)

where $erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-x^2}\, dx$. Thus, mathematically $\delta(x)$ is defined as

$$\delta(x) = \begin{Bmatrix} \infty & x = 0 \\ 0 & \text{otherwise} \end{Bmatrix}.$$

If we differentiate $\delta(x)$ to obtain a function $\delta'(x)$ with the property then

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x)F(x)\, dx &= \delta(x)F(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)F'(x)\, dx \\ &= -\int_{-\infty}^{\infty} \delta(x)F'(x)\, dx \\ &= -F'(0) \end{aligned} \tag{1.31}$$

for any continuous differentiable function $F(x)$. Behaviour like eqn (1.31) can again be realized in the limit of a sequence of functions (e.g. the derivatives of those in the sequence used to represent $\delta(x)$; these are depicted in graphical forms in Figure 1.4).

Physically, $\delta(x)$ can be regarded as that distribution of charge along the x-axis as a unit charge situated at the origin. Similarly, $\delta'(x)$ corresponds to a dipole of

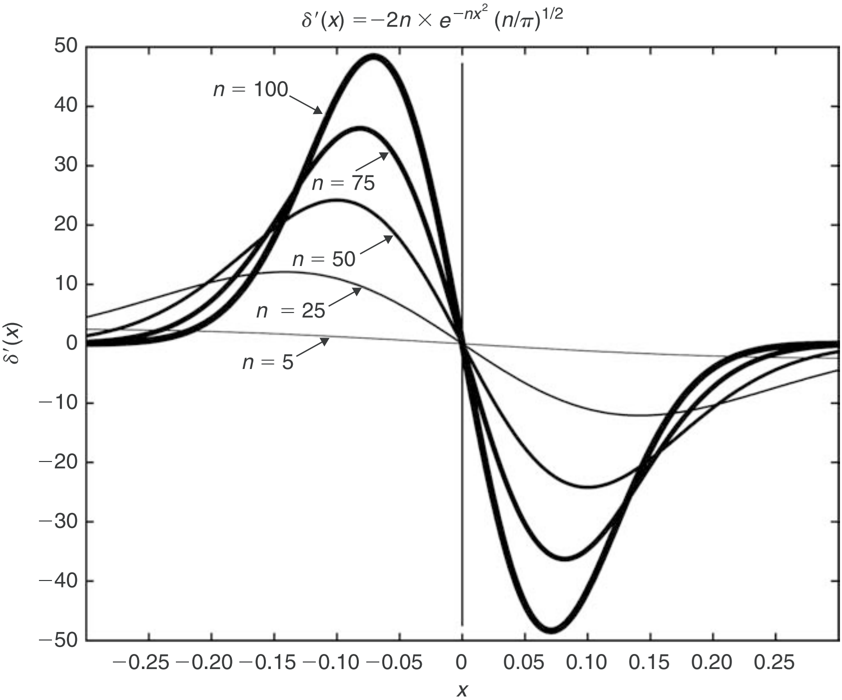

Figure 1.4: Functions in the sequence $\delta_n'(x) = -2nxe^{-nx^2}(\frac{n}{\pi})^{1/2}$ used to define $\delta'(x)$; the graphs are depicted for the number n (for $n = 5$, 25, 50, 75, 100) for the nth function. (From Lighthill, 1964.)

unit electric moment, since as a special case of eqn (1.31) we can obtain

$$\int_{-\infty}^{\infty} x\delta'(x)\,dx = x\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)\,dx = -1. \tag{1.32}$$

Thus, these generalized functions correspond to familiar physical idealization.

In defining generalized functions by means of sequences, we must define under what circumstances two sequences constitute the same generalized function. For this purpose, we need to multiply each member of a sequence by a "test function" $F(x)$, as in eqn (1.30) or (1.31), integrate from $-\infty$ to ∞, and take the limit. If the same result emerges for each sequence whatever "test function" is used, the sequences are said to define the same generalized function.

Although the delta function was the first generalized function to be introduced, the method of attaching values to integrals and series which are introduced in theory had much earlier forerunners, like Cauchy's "principal value", Hadamard's "finite part" of an improper integral and the theories of "summability" of series. These concepts are important with regard to the generalized function discussed and presented in this book. Lighthill's (1964) work is a summary of Temple's (1953, 1955) research papers and the works of Schwartz (1950–1951), Hadamard, Titchmarsh (1937) and Hardy & Rogosinski (1950). The reader is referred to their works for a better understanding about these concepts.

1.8 Heaviside unit step function and its relation with Dirac's delta function

Heaviside unit step function plays a very important role in the study of physical problems. Its importance is related to the definition of Dirac's delta function which was introduced in the last section. We define the Heaviside unit step function as follows:

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{1.33}$$

This definition implies that the value of the function is always unity when the argument is greater than zero, and its value is zero when the argument is always less than zero. The value of the function at the origin is not defined but can be assumed as the average of unity.

An impulse function which is singular at a certain point can be called the delta function, $\delta(x)$ (as we have seen in the last section). This function has a wide range of applications in many physical problems including signal processing. If the impulse occurs at a point $x = x_0$, then this function is defined as follows:

$$\delta(x - x_0) = \left\{ \begin{matrix} \infty & x = x_0 \\ 0 & \text{otherwise} \end{matrix} \right\},$$

which has the property $\int_{-\infty}^{\infty} \delta(x - x_0) F(x)\, dx = F(x_0) \int_{-\infty}^{\infty} \delta(x - x_0)\, dx = F(x_0)$, for any continuous function $F(x)$. The operation indicated involving the impulse function all across formally forms its integral definition. It is worth mentioning that the impulse function is not really a true function in the mathematical sense. The impulse function has, however, been justified mathematically using a theory of *generalized functions* as described by Lighthill (1964). With this approach, the impulse function is defined as the limit of a sequence of regular well-behaved functions which has the required property that the area remains constant (unity) as the width is reduced. Finally, the limit of this sequence is taken to define the impulse function as the width is reduced towards zero.

The defining sequence of pulses is not unique and many pulse shapes can be chosen. In fact, the shape of the particular pulse is relatively unimportant as long as the sequence satisfies the conditions that (i) the sequence formed describes a function which becomes infinitely high and infinitesimally narrow in such a way that (ii) the enclosed area is a constant (unity). For example, we select the following sequences all of which satisfy these conditions.

(a) Rectangular pulse: $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[H(x + \varepsilon/2) - H(x - \varepsilon/2)]$. This is the fundamental relationship between the Heaviside unit step function, $H(x)$, and the delta function, $\delta(x)$. We know that $[H(x+\varepsilon/2)-H(x-\varepsilon/2)]=1$ in the range $-\varepsilon/2<x<\varepsilon/2$, and outside this range it is zero. And hence when $\varepsilon\to 0$ the pulse becomes very tall while the width narrows down to zero. And also it is obvious that $\int_{-\varepsilon/2}^{\varepsilon/2}\delta(x)\,dx=1$ which implies that the property is completely satisfied. Thus, $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[H(x+\varepsilon/2)-H(x-\varepsilon/2)]$ is the true representation of $\delta(x)$.

(b) Triangular pulse: $\delta(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[1-\frac{|x|}{\varepsilon}\right]$, $|x|<\varepsilon$. This triangular pulse has the height $\frac{1}{\varepsilon}$, and the width of the pulse is 2ε. Hence the area of this pulse is equal to $\frac{1}{2}\times$ (height) $\times$ (width) $=\frac{1}{2}\left(\frac{1}{\varepsilon}\right)(2\varepsilon)=1$. It can be easily seen that when $\varepsilon\to 0$, the height of the pulse is infinite, and the width becomes zero. Hence it is a true representation of the delta function.

(c) Two-sided exponential pulse: $\delta(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}e^{-|2x|/\varepsilon}$. We shall integrate this sequence of functions with respect to x from $-\infty$ to ∞ to see whether or not the area covered by this curve is unity.

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(x)\,dx &= \int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}e^{-|2x|/\varepsilon}\,dx\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{-\infty}^{0}e^{2x/\varepsilon}\,dx+\int_{0}^{\infty}e^{-2x/\varepsilon}\,dx\right]\\ &= 1.\end{aligned}$$

Hence, it is a true representation of a delta function.

(d) Gaussian pulse: $\delta(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}e^{-\pi(x/\varepsilon)^2}$. We shall find the area covered by the sequence of these curves.

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(x)\,dx &= \int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}e^{-\pi(x/\varepsilon)^2}\,dx\\ &= \frac{2}{\sqrt{\pi}}\int_{0}^{\infty}e^{-\eta^2}\,d\eta\\ &= erf(\infty)\\ &= 1.\end{aligned}$$

Hence, it is a true representation of a delta function.

(e) The $Sa(x)$ function: $\delta(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}Sa(\frac{\pi x}{\varepsilon})$. This is another sequence of functions that determines the delta function. Let us assume that $X=\frac{\pi x}{\varepsilon}$. Then $Sa(X)=\frac{\sin(X)}{X}$. To determine the area covered by this function we calculate as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(x)\,dx &= \int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}Sa\left(\frac{\pi x}{\varepsilon}\right)dx\\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\left(\frac{1}{\varepsilon}\right)\left(\frac{\varepsilon}{\pi}\right)\left(\frac{\sin X}{X}\right)dX\\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin X}{X}\,dX\\ &= 1.\end{aligned}$$

Note that $\int_0^\infty \frac{\sin X}{X}\, dX = \frac{\pi}{2}$. Hence, it is a true representation of a delta function.

(f) The $Sa^2(x)$ function: $\delta(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} Sa^2\left(\frac{\pi x}{\varepsilon}\right)$. This sequence also determines the delta function. Proceeding exactly as before we obtain that

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x)\, dx &= \int_{-\infty}^{\infty} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} Sa\left(\frac{\pi x}{\varepsilon}\right) dx \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin^2 X}{X^2}\, dX \\ &= 1.\end{aligned}$$

This result is obtained by using the integration by parts. Hence, it is a true representation of a delta function.

We have illustrated a number of examples with regard to the sequence of functions that define Dirac's delta function and also its relationship with Heaviside unit step function. In the next chapter, the theoretical development of generalized functions and their Fourier transforms will be discussed with many examples.

1.9 Exercises

1. Find the complex form of the Fourier series of the periodic function whose definition in one period is

$$f(x) = e^x, \quad 0 \le x \le 1.$$

2. Find the Fourier series expansion of the periodic function whose definition in one period is

$$f(x) = x, \quad -1 < x < 1.$$

3. Find the Fourier cosine series expansion of the function

$$f(x) = x, \quad 0 < x < 1.$$

4. Find the half-range Fourier sine and cosine expansions of the function

$$f(x) = x(1+x), \quad 0 < x < 1.$$

5. Evaluate each of the following integrals:
   (a) $\int_{-1}^{1} e^{3x}\delta(x)\, dx$
   (b) $\int_{-1}^{1} e^{3x}\sin^2 x\delta''(x)\, dx$
   (c) $\int_0^2 x^2\delta'(x-1)\, dx$
   (d) $\int_{-\infty}^{\infty} \delta(\alpha - x)\delta(\alpha - y)\, d\alpha$
   (e) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha - x)\delta(\alpha - y)\delta(\beta - z)\, d\alpha\, d\beta$.

6. Prove that
   (a) $\int_{-\infty}^{\infty} \delta^{(m)}(\beta)\delta^{(n)}(x-\beta)\, d\beta = \delta^{(m+n)}(x)$
   (b) $\int_{-\infty}^{\infty} \delta(\beta - z)\delta(x-\beta)\, d\beta = \delta(x-z)$
   (c) $\int_{-\infty}^{\infty} \delta^{(m)}(\beta - y)\delta^{(n)}(x-\beta)\, d\beta = \delta^{(m+n)}(x-y)$.

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2 Generalized functions and their Fourier transforms

2.1 Introduction

This chapter contains some fundamental definitions and theorems which are vital for the development of generalized functions. We shall follow the concepts of Lighthill's (1964) work in manifesting the mathematics behind the theories. We shall illustrate with some examples the theory developed here. We follow Lighthill's mathematical definitions with the same mathematical symbols. The important development of the Heaviside unit step function and Dirac's delta function are clearly illustrated with figures, and the relationship between these two generalized functions is demonstrated. The definition of signum function is depicted by using figures and the relationship with the Heaviside unit function is given. Here we shall clearly explain the different terminologies with illustrations in a layman's term (see Rahman, 2001; Jones, 1982; Champeney, 1987).

2.2 Definitions of good functions and fairly good functions

Definition 1

A good function is an ordinary function which is differentiable any number of times and such that the function and all its derivatives are $O(|x|^{-N})$ as $|x| \to \infty$ for all N.

In a layman's term, a good function can be defined in a simple way as follows. Let us consider that $f(x)$ is a real or complex valued function of x for all real x and that $f(x)$ is everywhere continuous and infinitely differentiable and that each differential tends to zero as $x \to \pm\infty$ faster than any positive power of $\frac{1}{x}$, or in other words we can define it mathematically as follows:

$$\lim_{x \to \pm\infty} x^m f^{(n)}(x) = 0,$$

where m and n are both positive integers, then we say that $f(x)$ is a good function. For example, the functions defined by $f(x) = \exp(-x^2)$, $f(x) = x\exp(-x^2)$

and $f(x) = (1+x^2)^{-1}\exp(-x^2)$, respectively, are all good functions. However, the function $f(x) = \exp(-|x|)$ is not a good function because it is not differentiable at $x = 0$, and also the function $f(x) = \frac{1}{1+x^2}$ is not a good function because it decays too slowly as $x \to \pm\infty$.

Note that a good function has two properties in the sense that it decays rapidly at infinity and that it is differentiable infinitely many times. Good functions are also important because of the role they play in the theory of generalized functions. It is worth mentioning that good functions are not only continuous but are also uniformly and absolutely continuous on $(-\infty, \infty)$. However, a good function cannot necessarily be represented over every interval by a Taylor series expansion.

Definition 2

A fairly good function is one which is everywhere differentiable any number of times and such that the function and all its derivatives are $O(|x|^N)$ as $|x| \to \infty$ for all N.

Note that these two definitions look the same, but in fact it is not so. The difference between these two definitions is the exponent N in the order symbols. In the good function the exponent N is negative, whereas in the fairly good function the exponent N is positive. We shall illustrate this with some suitable examples.

Example 1

The exponential function $f(x) = e^{-x^2}$ is a good function.

Proof

It can be easily verified that this function is differentiable any number of times and such that the function and all its derivatives are $O(|x|^{-N})$ as $(|x| \to \infty)$ for all N. Because by Taylor's expansion we have

$$\begin{aligned} f(x) &= e^{-x^2} \\ &= \sum_{n=0}^{\infty} \left(\frac{x^{-2n}}{n!} \right) \\ &\sim O(|x|^{-N}) \text{ the } n\text{th order term.} \end{aligned}$$

Hence as $|x| \to \infty$ the function goes to zero for all positive N. It is a good function.

Example 2

Any polynomial function $P(x) = \sum_{n=0}^{N} a_n x^n$ is a fairly good function.

Proof

The nth order term is simply $O(|x|^N)$. This function can be differentiated as many times as we wish. So, it is a fairly good function.

Theorem 2.1

The derivative of a good function is a good function. The sum of two good functions is a good function. The product of a fairly good function and a good function is a good function.

Proof

Let us consider that the order of a good function $O(|x|)^{-N})$ and its derivative is $O(\frac{d}{dx}(|x|)^{-N}) = O(-N|x|^{-N-1}\, sgn(x)) = O(C|x|^{-M})$, where $M = N+1$, $C = -N\, sgn(x)$ and $\frac{d|x|}{dx} = sgn(x)$ in which

$$sgn(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases}$$

It is obvious that it is a good function. It is worth mentioning that $sgn(x)$ is a generalized function. It is worth noting that Heaviside unit step function is related to the signum function as $H(x) = \frac{1}{2} + \frac{1}{2}\, sgn(x)$. This function is also a generalized function.

The sum of two good functions is obviously a good function. The product of a good function and a fairly good function $O(|x|^{-N} \times |x|^{M} = |x|^{-N+M}) = O(|x|^{(-P)})$ is also a good function provided $P > 0$.

Theorem 2.2

If $f(x)$ is a good function, then Fourier transform of $f(x)$, namely

$$g(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx, \tag{2.1}$$

is a good function.

Proof

Differentiating eqn (2.1) p times with respect to y, and then integrating the resulting equation by parts N times, we can show that

$$\begin{aligned} |g^{(p)}(y)| &= \left| \frac{1}{(2\pi iy)^N} \int_{-\infty}^{\infty} \frac{d^N}{dx^N}\{(-2\pi ix)^p f(x)\} e^{-2\pi ixy}\, dx \right| \\ &\leq \frac{(2\pi)^{p-N}}{|y|^N} \int_{-\infty}^{\infty} \left| \frac{d^N}{dx^N}\{x^p f(x)\} \right| dx \\ &= O(|y|^{-N}), \end{aligned} \tag{2.2}$$

which proves the theorem.

Theorem 2.3

If $f(x)$ is a good function with its Fourier transform $g(y)$, then the Fourier transform of $f'(x)$ is $2\pi i y g(y)$, and the Fourier transform of $f(ax+b)$ is $|a|^{-1}e^{2\pi iby/a}g(y/a)$.

Proof

We know

$$\begin{aligned}
\mathcal{F}\{f'(x)\} &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi ixy}\,dx \\
&= -(-2\pi iy)\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\
&= (2\pi iy)g(y).
\end{aligned}$$

Next we show that

$$\begin{aligned}
\mathcal{F}\{f(ax+b)\} &= \int_{-\infty}^{\infty} f(ax+b)e^{-2\pi ixy}\,dx \\
&= \frac{1}{a}e^{2\pi iby/a}\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy/a}\,dx \\
&= \frac{1}{a}e^{2\pi iby/a}g(y/a) \\
&= \frac{1}{|a|}e^{2\pi iby/a}g(y/a),
\end{aligned}$$

which proves the theorem completely. $|a|$ emphasizes the fact that the coefficient of x must always be a positive number.

Theorem 2.4: Duality of Fourier transforms

If $g(y)$ is the Fourier transform of a good function $f(x)$, then $f(y)$ is the Fourier transform of $g(-x)$.

Proof

Let us consider the Fourier transform pairs

$$\begin{aligned}
g(y) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx, \\
f(x) &= \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy.
\end{aligned}$$

In the second integral, change x to y and then y to x which yields

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} g(x)e^{2\pi ixy}\,dx \\ &\quad \text{change } x \text{ to } -x \text{ under this integral sign} \\ &= \int_{-\infty}^{\infty} g(-x)e^{-2\pi ixy}\,dx \\ &= \mathcal{F}\{g(-x)\}. \end{aligned} \tag{2.3}$$

Thus Fourier transform of $g(-x)$ is $f(y)$. This theorem is known as the duality of Fourier transform. It has a wide range of applications in electrical engineering problems.

Theorem 2.5: Parseval's theorem of good functions

If $f_1(x)$ and $f_2(x)$ are good functions, and $g_1(y)$ and $g_2(y)$ are their Fourier transforms, then

$$\int_{-\infty}^{\infty} g_1(y)g_2(y)\,dy = \int_{-\infty}^{\infty} f_1(-x)f_2(x)\,dx. \tag{2.4}$$

Proof

The Fourier transforms of $f_1(x)$ and $f_2(x)$ are given by

$$\begin{aligned} g_1(y) &= \int_{-\infty}^{\infty} f_1(x)e^{-2\pi ixy}\,dx = \int_{-\infty}^{\infty} f_1(-x)e^{2\pi ixy}\,dx, \\ g_2(y) &= \int_{-\infty}^{\infty} f_2(x)e^{-2\pi ixy}\,dx. \end{aligned}$$

Hence we have

$$\begin{aligned} g_1(y)g_2(y) &= \int_{-\infty}^{\infty} f_1(-x)g_2(y)e^{2\pi ixy}\,dx \\ \int_{-\infty}^{\infty} g_1(y)g_2(y)\,dy &= \int_{-\infty}^{\infty} f_1(-x)\left\{\int_{-\infty}^{\infty} g_2(y)e^{2\pi ixy}\,dy\right\}dx \\ &= \int_{-\infty}^{\infty} f_1(-x)f_2(x)\,dx, \end{aligned}$$

which is the required proof of Theorem 2.5. This theorem has a wide range of applications in many applied fields. This theorem will also be used when $f_1(x)$ is a good function and $f_2(x)$ is any other function absolutely integrable from $-\infty$ to ∞. The proof stands word for word in this case, since the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(y)f_2(x)e^{-2\pi ixy}\,dx\,dy \tag{2.5}$$

remains absolutely convergent.

2.3 Generalized functions. The delta function and its derivatives

We have already defined the Dirac's delta function δ in Chapter 1. This function is the best known of a class of entities called *generalized function*. There are many more generalized functions that we will come across later in this book. The generalized functions are very important in Fourier transform theory because they allow the function to be Fourier transformed. In real sense, the function $f(x)=1$ has no Fourier transform, but it acquires the transform $\delta(x)$ in the generalized theory. Thus the generalized function removes a blockage which existed in the previous theory.

The Dirac delta function $\delta(x)$ is sometimes described as having the value zero for $x \neq 0$ and the value of infinity for $x = 0$. This is an extremely dangerous statement because it implies that a generalized function is specified by estimating its value for all, or almost all, values of x. In fact, they are specified in quite a different manner and are in reality very different entities from the regular functions.

The Dirac delta function $\delta(x)$ can be safely defined as the limit of a sequence of regular functions as illustrated in Section 1.7. For instance, we can construct a function as follows:

$$\delta(x) = \lim_{n\to\infty} n \exp(-\pi n^2 x^2).$$

It is worth noting here that to claim to be the delta function the area under this curve must be unity and this is true, because

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x)\, dx &= \lim_{n\to\infty} \int_{-\infty}^{\infty} n \exp(-\pi n^2 x^2)\, dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-y^2)\, dy = erf(\infty) = 1.\end{aligned}$$

This approach typifies one strand in the history of generalized functions, as illustrated by Lighthill (1964) and Jones (1982). Another definition of delta function is based on the idea that if a function is continuous at $x = 0$, then $\delta(x)$ is defined so that

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0),$$

or in general if the function is continuous at $x = x_0$, then

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0).$$

This approach manifests an alternative, and rather more formal, approach pioneered by Schwartz (1966), and employed in Gel'fand (1964–1969), Shilov (1968) and Zemanian (1965). In the following we will illustrate both approaches with examples.

Definition 3

A sequence $f_n(x)$ of good functions is called regular if, for any good function $F(x)$ whatever, the limit

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(x)\,dx \tag{2.6}$$

exists.

Example 3

Show that the sequence $f_n(x) = e^{-x^2/n^2}$ is regular.

Proof

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(x)\,dx &= \lim_{n\to\infty}\int_{-\infty} e^{-x^2/n^2}F(x)\,d'x\\ &= \int_{-\infty}^{\infty}\left\{\lim_{n\to\infty} e^{-x^2/n^2}\right\}F(x)\,dx\\ &= \int_{-\infty}^{\infty} F(x)\,dx.\end{aligned}$$

This is the required limit.

Definition 4

Two regular sequences of good functions are called equivalent if, for any good function $F(x)$ whatever, the limit (2.6) is the same for each sequence.

Example 4

Show that the sequence e^{-x^2/n^2} is equivalent to the sequence e^{-x^4/n^4}.

Proof

It is obvious that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n^2}F(x)\,dx = \int_{-\infty}^{\infty} F(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^4/n^4}F(x)\,dx.$$

Thus the given sequences are equivalent.

Definition 5

A generalized function $f(x)$ is defined as a regular sequence $f_n(x)$ of good functions, but two generalized functions are said to be equal if the corresponding regular sequences are equivalent. Thus each generalized function is really the class of all regular sequences equivalent to a given regular sequence.

The integral $\int_{-\infty}^{\infty} f(x)F(x)\,dx$ of the product of a generalized function $f(x)$ and a good function $F(x)$ is defined as

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(x)\,dx.$$

This is permissible because the limit is the same for all equivalent sequences $f_n(x)$.

Example 5

The sequence e^{-x^2/n^2} and all equivalent sequences define a generalized function $I(x)$ such that

$$\int_{-\infty}^{\infty} I(x)F(x)\,dx = \int_{-\infty}^{\infty} F(x)\,dx. \tag{2.7}$$

This generalized function $I(x)$ will be denoted simply by I.

Proof

Let us consider the sequence as $I_n(x) = e^{-x^2/n^2}$. Thus using Definition 5, we obtain

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} I_n(x)F(x)\,dx = \int_{-\infty}^{\infty}\left\{\lim_{n\to\infty} e^{-x^2/n^2}\right\}F(x)\,dx = \int_{-\infty}^{\infty} F(x)\,dx.$$

Hence the given sequence is a generalized function.

Example 6

The sequence equivalent to $e^{-nx^2}(n/\pi)^{1/2}$ defines a generalized function $\delta(x)$ such that

$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = F(0). \tag{2.8}$$

Proof

We know that if $F(x)$ is any good function then $\delta(x)F(x) = F(0)\delta(x)$. Hence $\int_{-\infty}^{\infty}\delta(x)F(x)\,dx = F(0)\int_{-\infty}^{\infty}\delta(x)\,dx = F(0)$. In a rigorous way we can prove that

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}F(x)\,dx - F(0)\right| &= \left|\int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}\{F(x) - F(0)\}\,dx\right| \\ &\le \max|F'(x)|\int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}|x|\,dx \\ &= (n\pi)^{-1/2}\max|F'(x)| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Remark

The sequence of function $I_n(x) = e^{-x^2/n^2}$ and the sequence of function $\delta_n(x) = e^{-nx^2}(n/\pi)^{1/2}$ are defined as $I(x)$ and $\delta(x)$ under the limiting condition as $n \to \infty$ such that $\lim_{n\to\infty} I_n(x) = I(x) = 1$ and $\lim_{n\to\infty} \delta_n(x) = \delta(x)$. $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that any constant functions including 1 and $\delta(x)$ are all generalized functions. It is clear that any Gaussian distribution function can be treated as a good function, whereas any polynomial function defines a fairly good function. It can be easily shown that $\int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}|x|\,dx = (n\pi)^{-1/2}$. And also, $\int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}\,dx = erf(\infty) = 1$.

Definition 6

If the generalized functions $f(x)$ and $h(x)$ are defined by sequences $f_n(x)$ and $h_n(x)$, then their sum $f(x) + h(x)$ is defined by the sequence $f_n(x) + h_n(x)$. Also the derivative $f'(x)$ is defined by the sequence $f_n'(x)$. Also, $f(ax+b)$ is defined by the sequence $f_n(ax+b)$. Also, $\phi(x)f(x)$, where $\phi(x)$ is a fairly good function, is defined by the sequence $\phi(x)f_n(x)$. Also, the Fourier transform $g(y)$ of $f(x)$ is defined by the sequence $g_n(y)$, where the Fourier transform of $f_n(x)$ is $g_n(y)$.

For proof of these definitions the reader is referred to Lighthill (1964). We do not want to reproduce his proofs again.

Example 7

Show that the Fourier transform of $\delta(x)$ is 1.

Proof

$$\begin{aligned} \mathcal{F}\{\delta(x)\} &= \int_{-\infty}^{\infty} \delta(x) e^{-2\pi ixy}\,dx \\ &= e^0 = 1. \end{aligned}$$

Hence the proof.

Example 8

Find the Fourier transform of $e^{-nx^2}(n/\pi)^{1/2}$. Then show that as n tends to infinity, the result becomes 1.

Solution

$$\begin{aligned}\mathcal{F}(e^{-nx^2}(n/\pi)^{1/2}) &= \int_{-\infty}^{\infty} e^{-nx^2}(n/\pi)^{1/2}e^{-2\pi ixy}\,dx\\ &= e^{-\pi^2y^2/n}\left[(n/\pi)^{1/2}\int_{-\infty}^{\infty} e^{-n(x+\pi iy/n)^2}\,dx\right]\\ &= e^{-\pi^2y^2/n}.\end{aligned}$$

It can be easily verified that $(n/\pi)^{1/2}\int_{-\infty}^{\infty} e^{-n(x+\pi iy/n)^2}\,dx = 1$.

Thus $\mathcal{F}(e^{-nx^2}(n/\pi)^{1/2}) = e^{-\pi^2y^2/n}$, which is obviously one of the sequences defining the generalized function 1.

Theorem 2.6

The following important results follow under the conditions of Definition 6 for any good function $F(x)$ with its Fourier transform $G(y)$:

$$\begin{aligned}\int_{-\infty}^{\infty} f'(x)F(x)\,dx &= -\int_{-\infty}^{\infty} f(x)F'(x)\,dx,\\ \int_{-\infty}^{\infty} f(ax+b)F(x)\,dx &= \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)F\left(\frac{x-b}{a}\right)\,dx,\\ \int_{-\infty}^{\infty}\{\phi(x)f(x)\}F(x)\,dx &= \int_{-\infty}^{\infty} f(x)\{\phi(x)F(x)\}\,dx,\\ \int_{-\infty}^{\infty} g(y)G(y)\,dy &= \int_{-\infty}^{\infty} f(x)F(-x)\,dx.\end{aligned} \tag{2.9}$$

The proofs of these results are obvious and the interested reader is referred to Lighthill (1964) and Rahman (2001).

Example 9

If $F(x)$ is any good function, then $\int_{-\infty}^{\infty}\delta^{(n)}(x)F(x)\,dx = (-1)^nF^{(n)}(0)$.

Proof

The proof will be given through induction using integration by parts.

$$\begin{aligned}\int_{-\infty}^{\infty}\delta^{(n)}(x)F(x)\,dx &= -\int_{-\infty}^{\infty}\delta^{(n-1)}(x)F'(x)\,dx\\ &= (-1)^2\int_{-\infty}^{\infty}\delta^{(n-2)}(x)F''(x)\,dx\end{aligned}$$

$$
\begin{aligned}
&= (-1)^3 \int_{-\infty}^{\infty} \delta^{(n-3)}(x) F^{(3)}(x)\, dx \\
&= (-1)^4 \int_{-\infty}^{\infty} \delta^{(n-4)}(x) F^{(4)}(x)\, dx \\
&\dots \\
&= (-1)^n \int_{-\infty}^{\infty} \delta(x) F^{(n)}(x)\, dx \\
&= (-1)^n F^{(n)}(0) \int_{-\infty}^{\infty} \delta(x)\, dx \\
&= (-1)^n F^{(n)}(0),
\end{aligned}
$$

which is the required proof.

Theorem 2.7

If $f(x)$ is a generalized function with Fourier transform $g(y)$, then the Fourier transform of $f(ax+b)$ is $\frac{1}{|a|}e^{2\pi iby/a}g(y)$. Also, Fourier transform of $f'(x)$ is $2\pi iyg(y)$. Finally, $f(y)$ is the Fourier transform of $g(-x)$.

Proof

The Fourier transform pair is

$$
\begin{aligned}
g(y) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx, \\
f(x) &= \int_{-\infty}^{\infty} g(y) e^{2\pi ixy}\, dy.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\mathcal{F}\{f(ax+b)\} &= \int_{-\infty}^{\infty} f(ax+b) e^{-2\pi ixy}\, dx \\
&= \int_{-\infty}^{\infty} f(z) e^{-2\pi iy(\frac{z-b}{a})}\, dz/a \\
&= \frac{1}{|a|} e^{2\pi iyb/a} \left\{ \int_{-\infty}^{\infty} f(z) e^{-2\pi izy/a}\, dz \right\} \\
&= \frac{1}{|a|} e^{2\pi iyb/a} g(y/a).
\end{aligned}
$$

Also, the Fourier transform of $f'(x)$ is

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi ixy}\,dx \\ &= 2\pi iy\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= 2\pi iyg(y).\end{aligned}$$

Also, from Fourier inversion we have

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy, \\ f(y) &= \int_{-\infty}^{\infty} g(x)e^{2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} g(-x)e^{-2\pi ixy}\,dx \\ &= \mathcal{F}\{g(-x)\}.\end{aligned}$$

Hence the proof.

Example 10

(a) Show that the Fourier transform of $\delta(x-c)$ is $e^{-2\pi icy}$. (b) Show that the Fourier transform of $e^{2\pi icx}$ is $\delta(y-c)$.

Proof

(a)

$$\begin{aligned}\mathcal{F}\{\delta(x-c)\} &= \int_{-\infty}^{\infty} \delta(x-c)e^{-2\pi ixy}\,dx \\ &= e^{-2\pi icy}.\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{F}\{e^{2\pi icx}\} &= \int_{-\infty}^{\infty} (e^{2\pi icx})(e^{-2\pi ixy})\,dx \\ &= \int_{-\infty}^{\infty} e^{-2\pi ix(y-c)}\,dx \\ &= \lim_{x\to\infty}\frac{\sin(2\pi x(y-c))}{\pi(y-c)} \\ &= \delta(y-c).\end{aligned}$$

Remark

Note that $\lim_{x\to\infty} \frac{\sin xy}{\pi y} = \delta(y)$. It can be easily shown that $\int_{-\infty}^{\infty} \delta(y-c)\,dy = \int_{-\infty}^{\infty} \delta(z)\,dz = 1$.

Example 11

Using the integral property of the delta function show that

$$\lim_{x\to\infty} \frac{\sin(2\pi x(y-c))}{\pi(y-c)} = \delta(y-c).$$

Proof

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(y-c)\,dy &= \int_{-\infty}^{\infty} \lim_{x\to\infty} \frac{\sin(2\pi x(y-c))}{\pi(y-c)}\,dy \\ &= \int_{-\infty}^{\infty} \lim_{x\to\infty} \frac{\sin(xz)}{\pi z}\,dz \\ &= \int_{-\infty}^{\infty} \delta(z)\,dz \\ &= 1.\end{aligned}$$

Hence this is the required proof.

We just cite two important theorems without proof (see Lighthill, 1964).

Theorem 2.8

If $f(x)$ is a generalized function and $f'(x) = 0$, then $f(x)$ is a constant, that is, $f(x)$ is equal to a constant times the generalized function 1.

Theorem 2.9

If $g(y)$ is a generalized function and $yg(y) = 0$, then $g(y)$ is a constant times $\delta(y)$.

It is worth noting here that

$$\lim_{n\to\infty} I_n(x) = e^{-x^2/n^2} = 1 = I(x) \quad \text{for any values of } x,$$

$$\lim_{n\to\infty} \delta_n(x) = e^{-nx^2}(n/\pi)^{1/2} = \delta(x) \quad \text{for any values of } x.$$

The relationship between $I_n(x)$ and $\delta_n(x)$ is given by

$$\frac{I_n(x)}{n\sqrt{\pi}} = \delta_n(x),$$

because $\int_{-\infty}^{\infty} \frac{I_n(x)}{n\sqrt{\pi}} dx = \int_{-\infty}^{\infty} \delta_n(x)\,dx$.

2.4 Ordinary functions as generalized functions

Definition 7

If $f(x)$ is a function of x in the ordinary sense, such that $(1+x^2)^{-N}f(x)$ is absolutely integrable from $-\infty$ to ∞ for some N, then the generalized function $f(x)$ is defined by a sequence $f_n(x)$ such that for any good function $F(x)$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)F(x)\,dx = \int_{-\infty}^{\infty} f(x)F(x)\,dx. \tag{2.10}$$

It is worth noting that the integral on the right-hand side is the integral in the ordinary sense, which exists as the integral of the product of $(1+x^2)^{-N}f(x)$, which is absolutely integrable, and $(1+x^2)^N F(x)$, which is a good function. When the generalized function $f(x)$ has been defined, this integral has a meaning also in the theory of generalized functions and eqn (2.10) states that these two meanings are the same. The consistency proof can be found in Lighthill (1964).

Example 12

Show that

$$\frac{d}{dx}(sgn(x)) = 2\delta(x).$$

Proof

The definitions of the generalized functions $H(x)$, $\delta(x)$ and $sgn(x)$ are graphically displayed in Figures 2.1 and 2.2. Mathematical definitions are as follows (Rahman, 2001):

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0; \end{cases}$$

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0; \end{cases}$$

$$sgn(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

We know that $\delta(x) = \lim_{\varepsilon\to 0}\frac{H(+\varepsilon/2)-H(x-\varepsilon/2)}{\varepsilon} = \frac{dH(x)}{dx}$. And also, we know $sgn(x) = 2H(x)-1$. Using these definitions, we obtain

$$\frac{d}{dx}(sgn(x)) = 2\frac{dH(x)}{dx} = 2\delta(x),$$

which is the required proof.

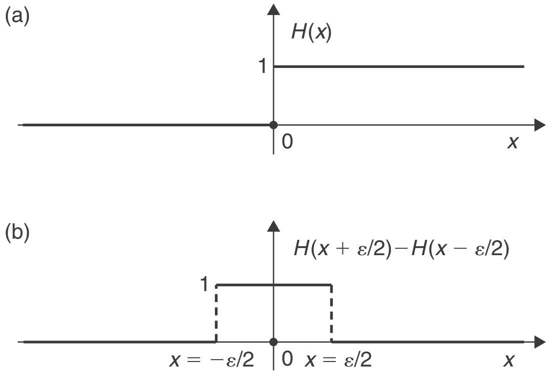

Figure 2.1: (a) Heaviside unit step function $H(x)$; (b) a window function.

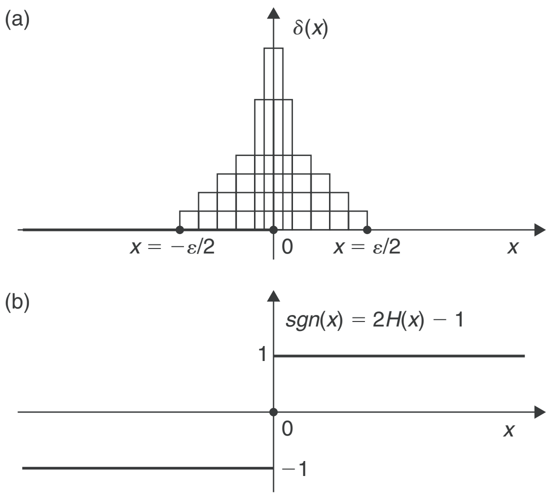

Figure 2.2: (a) Development of a delta function $\delta(x)$; (b) a signum function $sgn(x)$.

Theorem 2.10

If $f(x)$ is an ordinary differentiable function such that both $f(x)$ and $f'(x)$ satisfy the condition of Definition 7, then the derivative of the generalized function formed from $f(x)$ is the generalized function formed from $f'(x)$.

Theorem 2.11

If $f(x)$ is an ordinary function which is absolutely integrable from $-\infty$ to ∞, so that its Fourier transform $g(y)$ in the ordinary sense exists, then the Fourier transform of the generalized function $f(x)$ is the generalized function $g(y)$.

2.5 Equality of a generalized function and an ordinary function in an interval

Definition 8

If $h(x)$ is an ordinary function and $f(x)$ is a generalized function, and

$$\int_{-\infty}^{\infty} f(x)F(x)\,dx = \int_{a}^{b} h(x)F(x)\,dx \tag{2.11}$$

for every good function $F(x)$ which is zero outside $a < x < b$ (here a and b may be finite or infinite, and we assume the existence of the right-hand side of eqn (2.11) as an ordinary integral for all such $F(x)$, thus imposing a restriction on the function $h(x)$ in $a < x < b$, although it need not even be defined elsewhere), then we write

$$f(x) = h(x) \quad \text{for } a < x < b. \tag{2.12}$$

Example 13

Show that $\delta(x) = 0$ for $0 < x < \infty$ and $-\infty < x < 0$.

Proof

If $F(x)$ vanishes outside either of these two intervals, then $F(0) = 0$, and so by $\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = F(0)$, we have

$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = 0.$$

Theorem 2.12

If $h(x)$ and its derivatives $h'(x)$ are ordinary functions both satisfying the restriction imposed on $h(x)$ in Definition 8, and $f(x)$ is a generalized function which equals $h(x)$ in $a < x < b$, then

$$f'(x) = h'(x) \quad \text{in } a < x < b.$$

Example 14

Any repeated derivative $\delta^{(n)}(x)$ of the delta function equals 0 for $0 < x < \infty$ and for $-\infty < x < 0$ (by Theorem 2.12 and Example 13). It follows at once that any linear combination of $\delta^{(n)}(x)$ similarly vanishes everywhere except at $x = 0$, which is as interesting as showing what a wide variety of different generalized functions can all be equal at all points save one.

Example 15

If $f(x)$ and $g(x)$ are generalized functions such that $xf(x) = g(x)$, and if $g(x)$ equals an ordinary function $h(x)$ in the interval $a < x < b$ not including $x = 0$, then $f(x) = \frac{1}{x}h(x)$ in $a < x < b$.

Proof

If $F(x)$ is a good function which is zero outside $a < x < b$ then so is $\frac{1}{x}F(x)$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)F(x)\,dx &= \int_{-\infty}^{\infty} (xf(x))\left(\frac{1}{x}F(x)\right)dx\\ &= \int_{-\infty}^{\infty} g(x)\left(\frac{1}{x}F(x)\right)dx\\ &= \int_{a}^{b}\left(\frac{1}{x}h(x)\right)F(x)\,dx, \qquad (2.13)\end{aligned}$$

which proves that $f(x) = x^{-1}h(x)$.

2.6 Simple definition of even and odd generalized functions

Definition 9

The generalized function $f(x)$ is said to be even (or odd, respectively) if $\int_{-\infty}^{\infty} f(x)F(x)\,dx = 0$ for all odd (or even) good functions $F(x)$.

Example 16

Prove that $\delta(x)$ is an even generalized function.

Proof

$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = F(0) = 0.$$

This is true if and only if $F(x)$ is an odd good function.

Theorem 2.13

If the generalized function $f(x)$ is even (or odd, respectively), then its derivative $f'(x)$ is odd (or even), its Fourier transform $g(y)$ is even (or odd), while $\phi(x)f(x)$ is even (or odd) when the fairly good function $\phi(x)$ is even (or odd), and odd (or even) when $\phi(x)$ is odd (or even).

Example 17

$\delta^{(n)}(x)$ is even if n is even and odd if n is odd.

Proof

Let us consider that $F(x)$ is an odd good function. And we know that $\delta(x)$ is an even function. Also, we know that $\delta'(x)$ is an odd function.

$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = 0,$$
$$\int_{-\infty}^{\infty} \delta'(x)F(x)\,dx \neq 0,$$
$$\int_{-\infty}^{\infty} \delta''(x)F(x)\,dx = 0,$$
$$\int_{-\infty}^{\infty} \delta^{(3)}(x)F(x)\,dx \neq 0,$$
$$\int_{-\infty}^{\infty} \delta^{(4)}(x)F(x)\,dx = 0,$$
$$\cdots$$
$$\int_{-\infty}^{\infty} \delta^{(n)}(x)F(x)\,dx = 0 \quad \text{if } n \text{ is even},$$
$$\int_{-\infty}^{\infty} \delta^{(n)}(x)F(x)\,dx \neq 0 \quad \text{if } n \text{ is odd}.$$

Hence the required proof.

2.7 Rigorous definition of even and odd generalized functions

Definition 10

If $f_t(x)$ is a generalized function of x for each value of the parameter t and $f(x)$ is another generalized function, such that, for any good function $F(x)$,

$$\lim_{t\to c} \int_{-\infty}^{\infty} f_t(x)F(x)\,dx = \int_{-\infty}^{\infty} f(x)F(x)\,dx, \tag{2.14}$$

then we say

$$\lim_{t\to c} f_t(x) = f(x). \tag{2.15}$$

Example 18

Show that $\lim_{\varepsilon\to 0} \varepsilon|x|^{\varepsilon-1} = 2\delta(x)$.

Proof

This result can be obtained by differentiating the following relation:

$$\lim_{\varepsilon\to 0} |x|^{\varepsilon}\, sgn(x) = sgn(x).$$

Thus differentiating both sides with respect to x yields

$$\lim_{\varepsilon\to 0} \frac{d}{dx}(|x|^{\varepsilon}\, sgn(x)) = \frac{d}{dx}(sgn(x)),$$

$$\lim_{\varepsilon\to 0}\left[\varepsilon|x|^{\varepsilon-1}\, sgn^2(x) + |x|^{\varepsilon}\frac{d}{dx}(sgn(x))\right] = 2\delta(x),$$

$$\lim_{\varepsilon\to 0}\,[\varepsilon|x|^{\varepsilon-1} + 2|x|^{\varepsilon}\delta(x)] = 2\delta(x),$$

$$\lim_{\varepsilon\to 0}\ \varepsilon|x|^{\varepsilon-1} = 2\delta(x).$$

It is to be noted here that $sgn^2(x) = 1$ and $|x|^{\varepsilon}\delta(x) = 0$.

Example 19

Prove that

$$x^n\delta^{(m)}(x) = \begin{cases}(-1)^n\dfrac{m!}{(m-n)!}\delta^{(m-n)}(x) & (m \geq n),\\ 0 & (m < n).\end{cases}$$

Proof

This identity can be easily proved by the process of induction as illustrated below.

We know that $x\delta(x) = 0$. Now differentiating this equation with respect to x repeatedly up to m times yields the following:

$$\begin{aligned} x\delta(x) &= 0,\\ x\delta^{(1)}(x) &= -\delta(x),\\ x\delta^{(2)}(x) &= -2\delta^{(1)}(x),\\ x\delta^{(3)}(x) &= -3\delta^{(2)}(x),\\ &\cdots\\ x\delta^{(m)}(x) &= -m\delta^{(m-1)}(x).\end{aligned}$$

The last result can be expressed as follows:

$$x\delta^{(m)}(x) = \begin{cases}(-1)^1\dfrac{m!}{(m-1)!}\delta^{(m-1)}(x) & (m \geq 1),\\ 0 & (m < 1).\end{cases}$$

Now using the identity $x^2\delta(x) = 0$, and differentiating this equation with respect to x repeatedly up to m times and using the results obtained above yields the following:

$$\begin{aligned}
x^2\delta(x) &= 0,\\
x^2\delta^{(1)}(x) &= 0,\\
x^2\delta^{(2)}(x) &= 2\delta(x),\\
x^2\delta^{(3)}(x) &= 6\delta^{(1)}(x),\\
x^2\delta^{(4)}(x) &= 12\delta^{(2)}(x),\\
&\cdots\\
x^2\delta^{(m)}(x) &= m(m-1)\delta^{(m-2)}(x).
\end{aligned}$$

The last equation can be expressed easily as follows:

$$x^2\delta^{(m)}(x) = \begin{cases} (-1)^2\dfrac{m!}{(m-2)!}\delta^{(m-2)}(x) & (m \geq 2),\\ 0 & (m < 2). \end{cases}$$

Proceeding in this way we can easily write the identity $x^3\delta(x) = 0$ as

$$x^3\delta^{(m)}(x) = \begin{cases} (-1)^3\dfrac{m!}{(m-3)!}\delta^{(m-3)}(x) & (m \geq 3),\\ 0 & (m < 3). \end{cases}$$

Thus generalizing the result using the identity $x^n\delta(x) = 0$ and after repeated differentiation with respect to x, n times yields

$$x^n\delta^{(m)}(x) = \begin{cases} (-1)^n\dfrac{m!}{(m-n)!}\delta^{(m-n)}(x) & (m \geq n),\\ 0 & (m < n). \end{cases}$$

This completes the proof of this important identity.

Example 20

If $\phi(x)$ is any fairly good function, prove that

$$\phi(x)\delta(x) = \phi(0)\delta(x). \tag{2.16}$$

More generally, using the result of Example 19, or otherwise, show that

$$\phi(x)\delta^{(m)}(x) = \sum_{n=0}^{m}(-1)^n\frac{m!}{n!(m-n)!}\phi^{(n)}(0)\delta^{(m-n)}(x). \tag{2.17}$$

Proof

The function $\phi(x)$ can be expressed by Taylor's expansion about $x = 0$ up to mth order which yields

$$\phi(x) = \sum_{n=0}^{m} \phi^{(n)}(0)\frac{x^n}{n!}.$$

Hence we have after using the result of Example 19

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\delta^{(m)}(x)\,dx &= \int_{-\infty}^{\infty} \sum_{n=0}^{m} \phi^{(n)}(0)\frac{x^n}{n!}\delta^{(m)}(x)\,dx \\
&= \sum_{n=0}^{m} \frac{\phi^{(n)}(0)}{n!} \int_{-\infty}^{\infty} x^n \delta^{(m)}(x)\,dx \\
&= \sum_{n=0}^{m} (-1)^n \frac{m!}{n!(m-n)!}\phi^{(n)}(0) \int_{-\infty}^{\infty} \delta^{(m-n)}(x)\,dx \\
&= \int_{-\infty}^{\infty} \sum_{n=0}^{m} (-1)^n \frac{m!}{n!(m-n)!}\phi^{(n)}(0)\delta^{(m-n)}(x)\,dx,
\end{aligned}$$

which implies that

$$\phi(x)\delta^{(m)}(x) = \sum_{n=0}^{m} (-1)^n \frac{m!}{n!(m-n)!}\phi^{(n)}(0)\delta^{(m-n)}(x).$$

A very short-cut method to obtain this result can be stated as follows. Considering the expression for $\phi(x)$ and multiplying by $\delta^{(m)}(x)$ and then using the relation of Example 19 lead to the desired proof stated above.

Note here that if $\phi(x) = x^n$ then the above result can be written simply as

$$x^n \delta^{(m)}(x) = \begin{cases} (-1)^n \dfrac{m!}{(m-n)!}\delta^{(m-n)}(x) & (m \geq n), \\ 0 & (m < n). \end{cases}$$

This is exactly the same as Example 19.

Example 21

If $f(x)$ is a generalized function and $g(y)$ its Fourier transform, find the Fourier transform of $x^n f(x)$.

Solution

We know

$$g(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx.$$

Differentiating n times with respect to y

$$\begin{aligned} g^{(n)}(y) &= (-2\pi i)^n \int_{-\infty}^{\infty} (x^n f(x))e^{-2\pi ixy}\,dx \\ &= (-2\pi i)^n \mathcal{F}\{x^n f(x)\}. \end{aligned}$$

Thus rearranging the terms, we obtain

$$\mathcal{F}\{x^n f(x)\} = (-2\pi i)^{-n} g^{(n)}(y).$$

Example 22

Prove that

$$\lim_{n\to\infty} \frac{\sin nx}{\pi x} = \delta(x). \tag{2.18}$$

Proof

We know the integral property of a $\delta(x)$ function, that is, $\int_{-\infty}^{\infty}\delta(x)\,dx = 1$. Hence we have to prove that $\int_{-\infty}^{\infty}\lim_{n\to\infty}\frac{\sin nx}{\pi x}dx = 1$. Let us determine the value of the integral.

$$\begin{aligned} \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{\sin nx}{\pi x}dx &= \lim_{n\to\infty} \frac{2}{\pi}\int_0^{\infty} \frac{\sin t}{t}dt. \\ &\quad \text{This result is obtained by substituting } nx = t \\ &= \frac{2}{\pi} \times \left(\frac{\pi}{2}\right) \\ &= 1. \end{aligned}$$

Note that $\int_0^{\infty}\frac{\sin t}{t}dt = \frac{\pi}{2}$. This is the required proof.

2.8 Exercises

1. Determine which of the following functions are good function:
   (a) $f(x) = x^2 e^{-x^{2+\pi i/2}}$
   (b) $f(x) = e^{-x^{2+\pi i/2}} \sin 2x$
   (c) $f(x) = \tanh x - 1$

(d) $f(x) = e^{-|x|}$

(e) $f(x) = \frac{1}{x^3+1}$.

[Hint: A function $f(x)$ is said to be a good function if it is differentiable infinitely many times everywhere in the x–y plane and if

$$\lim_{|x|\to\infty} \left| x^p \frac{d^n f(x)}{dx^n} \right| = 0$$

for every integer $p \geq 0$ and every integer $n \geq 0$.]

2. Determine which of the following functions are fairly good function:

(a) $f(x) = e^{-x^2} \sin 2x$

(b) $f(x) = e^x$

(c) $f(x) = \frac{\cos x}{\cosh x}$

(d) $f(x) = (1 + x^2)^3$

(e) $f(x) = \tan 2x$.

[Hint: A function $f(x)$ is said to be a fairly good function if it is differentiable infinitely many times everywhere in the x–y plane and if there is some fixed N such that

$$\lim_{|x|\to\infty} \left| x^{-N} \frac{d^k f(x)}{dx^k} \right| = 0$$

for every integer $k \geq 0$.]

3. The function $f(x)$ is defined by

$$f(x) = \left\{ \begin{matrix} e^{-\frac{x^2}{(1-x^2)}} & |x| < 1 \\ 0 & |x| \geq 1 \end{matrix} \right\}.$$

The sequence $\{n_p\}$ is an increasing sequence of positive integers. Show that $\sum_{p=1}^{\infty} f(x - n_p)/n_p^p$ is a good function.

4. The function $U(x)$ is defined by

$$U(x) = \left\{ \begin{matrix} \int_{|x|}^{1} e^{-\frac{1}{t(1-t)}}\, dt \Big/ \int_{0}^{1} e^{-\frac{1}{t(1-t)}}\, dt & |x| < 1 \\ 0 & |x| \geq 1 \end{matrix} \right\}.$$

Show that $U(x) + U(x - 1) = 1$ when $0 \leq x \leq 1$ and deduce that $\sum_{m=-n}^{n} U(x - m)$ is a fine function which equals 1 for $|x| \leq n$.

[Hint: A function $U(x)$ is said to be fine if it is differentiable as many times as we wish (infinite number of times) on the metric space x–y and if the function and all its derivatives vanish identically outside some finite interval. Lighthill (1964) in his book defines this function as the unitary function.]

5. Find the Fourier transform of the following functions:

   (a) $f(x) = \operatorname{sech} \frac{1}{2}x$

   (b) $f(x) = (x^2 + 1)^{-1}$

   (c) $f(x) = (x^4 + 1)^{-1}$

   (d) $f(x) = xe^{-x^2/2}$

   (e) $f(x) = xe^{-(x-1)^2}$.

6. Prove that

$$\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2}}{x^2+1} dx = \sqrt{2\pi} \int_0^{\infty} e^{-(\alpha^2/2)-\alpha} \cos \alpha y \, d\alpha.$$

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3 Fourier transforms of particular generalized functions

3.1 Introduction

This chapter deals with a number of particular generalized functions that are derived and studied, some for their intrinsic interest and widespread utility, and others solely for their applications to techniques of asymptotic estimation. The following four sections deal with four important topics, namely, generalized functions with non-integral powers, non-integral powers multiplied by logarithms, integral powers and integral powers multiplied by logarithms. A brief summary of the Fourier transforms of some important generalized functions is presented at the end of the chapter.

Extensive tabulations of Fourier transforms of functions can be found, for instance, in Gel'fand (1964–1969) and Jones (1982). Although these authors show that an extension to complex powers of x is possible, we will restrict ourselves to real powers. The gamma functions $\Gamma(\alpha)$, where α can also be a fraction, and $\Gamma(m) = (m-1)!$, where m is an integer, are tabulated for instance in Abramowitz & Stegun (1966) whilst the function $\psi(m)$ is defined in Watson (1944) and Champeney (1987).

We now begin by defining non-integral powers of some important generalized functions. The reader is referred to the following works: Temple (1953, 1955), Lighthill (1964), Rahman (2001) and Jones (1982).

3.2 Non-integral powers

The following are the three most interesting generalized functions with non-integral powers (α):

$$|x|^{\alpha}, \quad |x|^{\alpha}\, sgn(x) \quad \text{and} \quad x^{\alpha}H(x) = \frac{1}{2}\{|x|^{\alpha} + |x|^{\alpha}\, sgn(x)\}, \tag{3.1}$$

where $H(x)$ is the Heaviside unit step function as defined in Figure 2.1(a). The first expression in eqn (3.1) is an even function, the second odd and the third (the mean of the other two) vanishes for $x < 0$.

These expressions are by definition the generalized functions of x, where $\alpha > -1$. The derivatives of these expressions with respect to x are as follows for $\alpha > 0$:

$$
\begin{aligned}
\frac{d}{dx}(|x|^\alpha) &= \alpha|x|^{\alpha-1}\frac{d|x|}{dx} = \alpha|x|^{\alpha-1}\, sgn(x), \\
\frac{d}{dx}(|x|^\alpha\, sgn(x)) &= \alpha|x|^{\alpha-1}\, sgn^2(x) + |x|^\alpha \frac{d}{dx}(sgn(x)) = \alpha|x|^{\alpha-1}, \\
\frac{d}{dx}(x^\alpha H(x)) &= \alpha x^{\alpha-1}H(x) + x^\alpha \frac{dH(x)}{dx} = \alpha x^{\alpha-1}H(x).
\end{aligned} \tag{3.2}
$$

Note that $|x|^\alpha \frac{d}{dx}(sgn(x)) = |x|^\alpha(2\delta(x)) = 0$ and $x^\alpha \frac{dH(x)}{dx} = x^\alpha\delta(x) = 0$. Also note that $\frac{d}{dx}(sgn(x)) = 2\delta(x)$ and $\frac{H(x)}{dx} = \delta(x)$. It is convenient to use these equations for $\alpha < 0$, repeatedly if necessary, to define the appropriate generalized functions for non-integral $\alpha < 0$.

Definition 1

If $\alpha < -1$ and is not an integer, then we define three new generalized functions as follows:

$$|x|^\alpha = \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\frac{d^n}{dx^n}\{|x|^{\alpha+n}(sgn\, x)^n\}, \tag{3.3}$$

$$|x|^\alpha\, sgn(x) = \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\frac{d^n}{dx^n}\{|x|^{\alpha+n}(sgn\, x)^{n+1}\}, \tag{3.4}$$

$$x^\alpha H(x) = \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\frac{d^n}{dx^n}\{x^{\alpha+n}H(x)\}, \tag{3.5}$$

where n is an integer such that $\alpha + n > -1$. These results can easily be proved by differentiating with respect to x on the right-hand side repeatedly. Definition 1 now extends the validity of these equations to all non-integral α. The relation (3.1) between the three functions, known in the first place for $\alpha > -1$, is similarly extended, by repeated differentiation, to the three new generalized functions defined by eqns (3.3), (3.4) and (3.5). It is worth noting that $|x|^\alpha$ is an even and $|x|^\alpha\, sgn(x)$ an odd generalized function.

We can use Definition 1 to interpret "improper" integrals like $\int_0^a x^\alpha F(x)\, dx$, in which α is non-integral and < -1 and $F(x)$ is a good function, as

$$\int_0^a x^\alpha F(x)\, dx = \int_{-\infty}^{\infty} x^\alpha [H(x) - H(x-a)]F(x)\, dx, \tag{3.6}$$

where $[H(x) - H(x-a)]$ is a window function which is equal to 1 in the interval $0 < x < a$, and can be treated as a generalized function. Integration by parts can be used to express eqn (3.6) as an ordinary integral. However, we need the fact that, if $f(x)$ is an ordinary function (here x^α) differentiable for $x \geq a$, then

$$\begin{aligned}\frac{d}{dx}[f(x)H(x-a)] &= \frac{d}{dx}[(f(x)-f(a))H(x-a)] + f(a)\delta(x-a)\\ &= f'(x)H(x-a) + f(a)\delta(x-a), \qquad (3.7)\end{aligned}$$

which is easy to remember because of the simplicity of the ordinary rule for differentiating a product. Applying this repeatedly to the function $x^\alpha H(x-a)$, we can obtain

$$\begin{aligned}\frac{d}{dx}[x^{\alpha+1}H(x-a)] &= (\alpha+1)x^\alpha H(x-a) + x^{\alpha+1}\frac{d}{dx}(H(x-a))\\ &= (\alpha+1)x^\alpha H(x-a) + a^{\alpha+1}\delta(x-a),\\ \frac{d^2}{dx^2}[x^{\alpha+2}H(x-a)] &= \frac{d}{dx}[(\alpha+2)x^{\alpha+1}H(x-a) + a^{\alpha+2}\delta(x-a)]\\ &= (\alpha+1)(\alpha+2)x^\alpha H(x-a) + (\alpha+2)a^{\alpha+1}\delta(x-a)\\ &\quad + a^{\alpha+2}\delta'(x-a)\\ &= (\alpha+1)(\alpha+2)\left[x^\alpha H(x-a) + \frac{a^{\alpha+1}}{\alpha+1}\delta(x-a)\right.\\ &\qquad \left. + \frac{a^{\alpha+2}}{(\alpha+1)(\alpha+2)}\delta'(x-a)\right].\end{aligned}$$

Thus, rearranging the terms we have

$$\begin{aligned}x^\alpha H(x-a) &= \frac{1}{(\alpha+1)}\frac{d}{dx}[x^{\alpha+1}H(x-a)] - \frac{a^{\alpha+1}}{(\alpha+1)}\delta(x-a)\\ &= \frac{1}{(\alpha+1)(\alpha+2)}\frac{d^2}{dx^2}[x^{\alpha+2}H(x-a)] - \frac{a^{\alpha+1}}{(\alpha+1)}\delta(x-a)\\ &\quad - \frac{a^{\alpha+2}}{(\alpha+1)(\alpha+2)}\delta'(x-a).\end{aligned}$$

Thus, with this information, we can write the following differentiation rule:

$$\begin{aligned}x^\alpha[H(x)-H(x-a)] &= \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\frac{d^n}{dx^n}[x^{\alpha+n}\{H(x)-H(x-a)\}]\\ &\quad + \frac{a^{\alpha+1}}{\alpha+1}\delta(x-a) + \frac{a^{\alpha+2}}{(\alpha+1)(\alpha+2)}\delta'(x-a)\\ &\quad + \cdots + \frac{a^{\alpha+n}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\delta^{(n-1)}(x-a). \qquad (3.8)\end{aligned}$$

Let us now consider the "improper" integral $\int_0^a x^\alpha F(x)\,dx$, where α is non-integer and <-1. The integration can be performed by the ordinary formula for repeated integration by parts.

$$\int_0^a x^\alpha F(x)\,dx = \frac{(-1)^n}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\int_0^a x^{\alpha+n}F^{(n)}(x)\,dx + \frac{a^{\alpha+1}}{\alpha+1}F(a) - \frac{a^{\alpha+2}}{(\alpha+1)(\alpha+2)}F'(a) + \cdots + \frac{(-1)^{n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}F^{(n-1)}(a). \tag{3.9}$$

In evaluating this integral we have omitted all the contributions from the lower limit which involve $0^{\alpha+1}, 0^{\alpha+2}, \ldots, 0^{\alpha+n}$, and so are infinite if n is the least integer with $\alpha+n>-1$. This concludes the discussion about differentiation and integration rule.

Let us now concentrate on finding the Fourier transforms of the functions defined in Definition 1. To proceed with this, we need the definition of a gamma function $\Gamma(\alpha+1)=\alpha!$. The integral definition for $\alpha>-1$ is

$$\int_0^\infty x^\alpha e^{-x}\,dx = \Gamma(\alpha+1) = \alpha!, \tag{3.10}$$

where x^α is of course positive. The straight path of integration in eqn (3.10) may be rotated about the origin through any angle $<\frac{\pi}{2}$, provided that x^α is taken as a function regular in the right-hand half plane which is equal to the positive number x^α for real positive x; in other words, the many-valued function x^α must be made precise by taking $|\arg x|<\pi/2$. Hence, if a is any complex number with positive real part,

$$\int_0^\infty x^\alpha e^{-ax}\,dx = a^{-\alpha-1}\int_0^\infty z^\alpha e^{-z}\,dz = a^{-\alpha-1}\Gamma(\alpha+1) = a^{-\alpha-1}\alpha!, \tag{3.11}$$

by the substitution $ax=z$; and in eqn (3.11), we must have $|\arg a|<\pi/2$, in order that $\arg z=\arg a$ shall satisfy the same condition.

In the following we shall demonstrate how to obtain the Fourier transforms of some generalized functions by using the result in eqn (3.11).

Example 1

Find the Fourier transform of $x^\alpha H(x)$ where α is non-integer and ≥ -1.

Solution

To obtain the Fourier transform of $x^\alpha H(x)$ for $\alpha\geq-1$, we use the limit property

$$\lim_{t\to 0} x^\alpha e^{-tx}H(x) = x^\alpha H(x), \tag{3.12}$$

where t is just a parameter introduced to ease the integration.

$$\begin{aligned}
\mathcal{F}\{x^\alpha H(x)\} &= \lim_{t\to 0} \mathcal{F}\{x^\alpha e^{-tx} H(x)\} \\
&= \lim_{t\to 0} \int_0^\infty x^\alpha e^{-tx} e^{-2\pi i x y}\, dx \\
&= \lim_{t\to 0} \int_0^\infty x^\alpha e^{-(t+2\pi i y)x}\, dx \\
&= \lim_{t\to 0} \alpha!(t + 2\pi i y)^{-(\alpha+1)} \\
&= \alpha!(2\pi i y)^{-(\alpha+1)}.
\end{aligned}$$

Now we know that $i^{-(\alpha+1)} = e^{-\pi i/2(\alpha+1)}$, and hence using this relation we have

$$\mathcal{F}\{x^\alpha H(x)\} = [e^{-\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)}.$$

Example 2

If Fourier transform of $f(x)$ is $g(y)$ then Fourier transform of $f'(x)$ is $2\pi i y g(y)$. Using this relation determine the Fourier transform of $\alpha x^{\alpha-1} H(x)$.

Solution

From Example 1 we know $\mathcal{F}\{x^\alpha H(x)\} = [e^{-\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)}$. Therefore, the Fourier transform of the derivative of $x^\alpha H(x)$ which is $\alpha x^{\alpha-1} H(x)$ must be $(2\pi i y) \times [e^{-\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)} = [e^{-\pi i/2\alpha\, sgn(y)}]\alpha!(2\pi|y|)^{-\alpha}$, since $iy = e^{\pi i/2\, sgn(y)}|y|$. We must make sure that y is positive absolute.

Example 3

Determine the Fourier transform of $|x|^\alpha$.

Solution

Since $|x|^\alpha = x^\alpha H(x) + (-x)^\alpha H(-x)$, the Fourier transform can be written as

$$\begin{aligned}
\mathcal{F}\{|x|^\alpha\} &= \mathcal{F}\{x^\alpha H(x)\} + \mathcal{F}\{(-x)^\alpha H(-x)\} \\
&= [e^{-\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)} + [e^{\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)} \\
&= (2\cos \pi/2(\alpha+1))\alpha!(2\pi|y|)^{-(\alpha+1)},
\end{aligned}$$

which is an even function.

Example 4

Determine the Fourier transform of $|x|^\alpha\, sgn(x)$.

Solution

Since $|x|^{\alpha}\, sgn(x) = x^{\alpha}H(x) - (-x)^{\alpha}H(-x)$, the Fourier transform can be written as

$$\begin{aligned}\mathcal{F}\{|x|^{\alpha}\}\, sgn(x) &= \mathcal{F}\{x^{\alpha}H(x)\} - \mathcal{F}\{(-x)^{\alpha}H(-x)\}\\ &= [e^{-\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)}\\ &\quad - [e^{\pi i/2(\alpha+1)\, sgn(y)}]\alpha!(2\pi|y|)^{-(\alpha+1)}\\ &= (-2i \sin \pi/2(\alpha+1))\alpha!(2\pi|y|)^{-(\alpha+1)}\, sgn(y),\end{aligned}$$

which is an odd function.

Example 5

Evaluate the integral $\int_0^1 \frac{x^{-1/2}}{1+x}dx$.

Solution

The solution can be easily obtained by substituting $x = \tan^2\theta$ such that $dx = 2\tan\theta \sec^2\theta\, d\theta$, and after a little reduction we obtain

$$\begin{aligned}\int_0^1 \frac{x^{-1/2}}{1+x}\, dx &= 2\int_0^{\pi/4} d\theta\\ &= 2\times\frac{\pi}{4} = \frac{\pi}{2}.\end{aligned}$$

Example 6

Evaluate the integral

$$\int_0^1 \frac{x^{-3/2}}{1+x}\, dx.$$

Solution

The solution can be obtained by using integration by parts and then substituting $x = \tan^2\theta$ such that $dx = 2\tan\theta \sec^2\theta\, d\theta$.

$$\begin{aligned}\int_0^1 \frac{x^{-3/2}}{1+x}\, dx &= -4\int_0^{\pi/4} \cos^2\theta\, d\theta\\ &= -1 - \frac{\pi}{2}\\ &= \left|1 + \frac{\pi}{2}\right|.\end{aligned}$$

It is worth noting here that the modulus sign is essential for the value of the integral because of the branches of the given function. If we plot the function it is obvious that the area covered by the curve must be a positive number.

Example 7

Show that

$$\int_0^1 \frac{x^{-5/2}}{1+x}dx = \frac{\pi}{2} + \frac{4}{3}.$$

Proof

This integral can be evaluated by using the method of integration by parts and we have given only the highlights of different steps leading to the answer.

$$\begin{aligned}\int_0^1 \frac{x^{-5/2}}{1+x}dx &= -\frac{1}{3} + \frac{1}{3} + \frac{8}{3}\int_0^1 \frac{x^{-1/2}}{1+x}dx \\ &= \frac{8}{3}\int_0^1 \frac{x^{-1/2}}{1+x}dx.\end{aligned}$$

This integral can be evaluated very easily by substituting $x = \tan^2\theta$. Hence displaying only the main steps, we have

$$\begin{aligned}\int_0^1 \frac{x^{-5/2}}{1+x}dx &= \frac{16}{3}\int_0^{\pi/4} \cos^4\theta\, d\theta \\ &= \frac{4}{3}\left[1 + \frac{3\pi}{8}\right] \\ &= \frac{4}{3} + \frac{\pi}{2}.\end{aligned}$$

It is worth noting in evaluating this integral that the values at the lower limit such as $0^{-3/2}$ and $0^{-1/2}$ tend to *infinity* and these values are ignored.

Example 8

Prove that the equation $xf(x) = |x|^\alpha$ is satisfied by $f(x) = |x|^{\alpha-1}\, sgn(x)$. Are there any other solutions?

Proof

Given that

$$xf(x) = |x|^\alpha,$$

we have

$$\begin{aligned}f(x) &= \left(\frac{|x|^\alpha}{|x|}\right)\left(\frac{|x|}{x}\right) \\ &= |x|^{\alpha-1} sgn(x).\end{aligned}$$

Example 9

Check that Fourier's inversion theorem is satisfied by both $|x|^\alpha$ and $|x|^\alpha\, sgn(x)$.

Solution

We know if $\mathcal{F}\{f(x)\} = g(y)$ then by using the duality property of the Fourier transform we have $\mathcal{F}\{g(x)\} = f(-y)$. Thus, we have
(a)

$$\mathcal{F}\{|x|^\alpha\} = (2\cos \pi/2(\alpha+1))\,\alpha!(2\pi|y|)^{-(\alpha+1)}$$

and hence

$$\mathcal{F}\{|x|^{-(\alpha+1)}\} = \frac{(2\pi)^{\alpha+1}|-y|^\alpha}{(2\cos \pi/2(\alpha+1))\,\alpha!};$$

(b)

$$\mathcal{F}\{|x|^\alpha\, sgn(x)\} = (-2i\sin \pi/2(\alpha+1))\,\alpha!(2\pi|y|)^{-(\alpha+1)}\, sgn(y)$$

and hence

$$\mathcal{F}\{|x|^{-(\alpha+1)}\, sgn(x)\} = \frac{(2\pi)^{\alpha+1}|-y|^\alpha\, sgn(y)}{(-2i\sin \pi/2(\alpha+1))\,\alpha!}.$$

3.3 Non-integral powers multiplied by logarithms

We again consider the three generalized functions that we have defined in the last section, where we have also demonstrated the way to find the Fourier transforms. In this section we consider them with non-integral powers multiplied by logarithms.

Definition 2

$$\begin{aligned}
|x|^\alpha \ln|x| &= \frac{\partial}{\partial\alpha}|x|^\alpha,\\
|x|^\alpha \ln|x|\, sgn(x) &= \frac{\partial}{\partial\alpha}(|x|^\alpha sgn(x)),\\
|x|^\alpha \ln|x| H(x) &= \frac{\partial}{\partial\alpha}(|x|^\alpha H(x)).
\end{aligned} \tag{3.13}$$

They can be easily proved. We know $|x|^\alpha = e^{\alpha\ln|x|}$ and so differentiating with respect to α partially we obtain the desired result.

Remark

These equations are of course true in the ordinary sense for $\alpha > -1$. For non-integral $\alpha < -1$, they must be taken as defining the generalized functions on the left-hand side; the derivatives with respect to α then exist by repeated application. The ordinary rule of differentiation applies to these functions; for example, we obtain

$$\begin{aligned}\frac{d}{dx}\{|x|^\alpha \ln|x|\} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \alpha}|x|^\alpha\right)\\ &= \frac{\partial}{\partial \alpha}\left(\frac{\partial}{\partial x}|x|^\alpha\right)\\ &= \frac{\partial}{\partial \alpha}\left[\alpha|x|^{\alpha-1}\frac{d|x|}{dx}\right]\\ &= \frac{\partial}{\partial \alpha}[\alpha|x|^{\alpha-1}\, sgn(x)]\\ &= |x|^{\alpha-1}\, sgn(x) + \alpha|x|^{\alpha-1}\ln|x|\, sgn(x). \end{aligned} \tag{3.14}$$

Now we shall obtain the Fourier transforms of three important generalized functions as defined above.

$$\begin{aligned}\mathcal{F}\{|x|^\alpha \ln|x|\} &= \mathcal{F}\left\{\frac{\partial}{\partial \alpha}|x|^\alpha\right\}\\ &= \frac{\partial}{\partial \alpha}\mathcal{F}\{|x|^\alpha\}\\ &= \frac{\partial}{\partial \alpha}\{(2\cos \pi/2(\alpha+1))\,\alpha!(2\pi|y|)^{-\alpha-1}\}\\ &= \Big[2\cos \pi/2(\alpha+1)\,\alpha!(2\pi|y|)^{-\alpha-1}\\ &\quad\times\left(-\ln(2\pi|y|) + \psi(\alpha) - \frac{\pi}{2}\tan\frac{\pi}{2}(\alpha+1)\right)\Big], \end{aligned} \tag{3.15}$$

where $\psi(\alpha) = \frac{d}{d\alpha}(\ln \alpha!)$.

Similarly the Fourier transform of $|x|^\alpha \ln|x|\, sgn(x)$ is given by

$$\begin{aligned}\mathcal{F}\{|x|^\alpha \ln|x|\, sgn(x)\} &= \mathcal{F}\left\{\frac{\partial}{\partial \alpha}|x|^\alpha\, sgn(x)\right\}\\ &= \frac{\partial}{\partial \alpha}\mathcal{F}\{|x|^\alpha\, sgn(x)\}\\ &= \frac{\partial}{\partial \alpha}\{(-2i\sin \pi/2(\alpha+1))\,\alpha!(2\pi|y|)^{-\alpha-1}\, sgn(x)\}\\ &= \Big[-2i\sin \pi/2(\alpha+1)\,\alpha!(2\pi|y|)^{-\alpha-1}\, sgn(x)\\ &\quad\times\left(-\ln(2\pi|y|) + \psi(\alpha) + \frac{\pi}{2}\cot\frac{\pi}{2}(\alpha+1)\right)\Big]. \end{aligned} \tag{3.16}$$

Also we obtain the Fourier transform of $x^\alpha \ln x\, H(x)$:

$$\begin{aligned}\mathcal{F}\{x^\alpha \ln|x| H(x)\} &= \mathcal{F}\left\{\frac{\partial}{\partial\alpha} x^\alpha H(x)\right\} \\ &= \frac{\partial}{\partial\alpha}\mathcal{F}\{x^\alpha H(x)\} \\ &= \frac{\partial}{\partial\alpha}\{e^{-\frac{\pi i}{2}(\alpha+1)\, sgn(y)}\alpha!(2\pi|y|)^{-\alpha-1}\} \\ &= \{e^{-\frac{\pi i}{2}(\alpha+1)\, sgn(y)}\alpha!(2\pi|y|)^{-\alpha-1}\} \\ &\quad \times \left[-\frac{\pi i}{2} sgn(y) + \psi(\alpha) - \ln(2\pi|y|)\right]. \end{aligned} \tag{3.17}$$

We next turn our attention to the integral powers of the functions to study their Fourier transforms.

3.4 Integral powers of an algebraic function

We consider here two important parameters: n signifies any integer ≥ 0 and m any integer >0. We deduce one important relation: if $f(x)$ has the Fourier transform $g(y)$ then

$$\mathcal{F}\{x^n f(x)\} = (-2\pi i)^{-n} g^{(n)}(y).$$

It can be proved as follows. We know

$$g(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx,$$

and hence differentiating n times with respect to y

$$g^{(n)}(y) = (-2\pi i)^n \int_{-\infty}^{\infty} x^n f(x) e^{-2\pi ixy}\, dx$$

or rearranging the terms, we obtain

$$\mathcal{F}\{x^n f(x)\} = (-2\pi i)^{-n} g^{(n)}(y). \tag{3.18}$$

Example 10

Find the Fourier transform of x^n.

Solution

We know $\mathcal{F}(1) = \int_{-\infty}^{\infty} (1) e^{-2\pi ixy}\, dx = \delta(y)$. That means

$$\delta(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy}\, dx.$$

Differentiating both sides n times with respect to y, and then transposing the result as before, we obtain

$$\mathcal{F}\{x^n\} = (-2\pi i)^{(-n)}\delta^{(n)}(y).$$

This is the required result. Lighthill (1964, p. 36) has shown the result in a rigorous way.

Example 11

Show that the Fourier transform of $x^n\, sgn(x)$ is $2(n!)(2\pi iy)^{-n-1}$.

Proof

$$\begin{aligned}
\mathcal{F}\{x^n\, sgn(x)\} &= \int_{-\infty}^{\infty} (x^n\, sgn(x))e^{-2\pi ixy}\,dx \\
&= \int_0^{\infty} x^n e^{-2\pi ixy}\,dx - \int_{-\infty}^{0} x^n e^{-2\pi ixy}\,dx \\
&= \int_0^{\infty} x^n e^{-2\pi ixy}\,dx + (-1)^{n+1}\int_0^{\infty} x^n e^{2\pi ixy}\,dx \\
&\quad \text{By using the formula for gamma function, we obtain} \\
&= n!(2\pi iy)^{-n-1} + (-1)^{n+1} n!(-2\pi ixy)^{-n-1} \\
&= 2(n!)(2\pi ixy)^{-n-1}.
\end{aligned}$$

This is the required proof.

Example 12

Find the Fourier transform of $sgn(x)$.

Solution

The signum function has an average value of zero and is piecewise continuous, but not absolutely integrable. In order to make it absolutely integrable, we multiply $sgn(x)$ by $e^{-a|x|}$ and then take the limit as $a \to 0$.

$$\begin{aligned}
\mathcal{F}\{sgn(x)\} &= \mathcal{F}\{\lim_{a\to 0}\,[e^{-a|x|}\, sgn(x)]\} \\
&\quad \text{Interchanging the operations of taking the limit and integrating} \\
&= \lim_{a\to 0}\left[\int_{-\infty}^{\infty} e^{-a|x|}\, sgn(x)e^{-2\pi ixy}\,dx\right]
\end{aligned}$$

$$= \lim_{a\to 0}\left[-\int_{-\infty}^{0} e^{(a-2\pi iy)x}\,dx + \int_{0}^{\infty} e^{-(a+2\pi iy)x}\,dx\right]$$

Performing these integrations, we obtain

$$= \lim_{a\to 0}\left[\frac{-4\pi iy}{a^2+4\pi^2y^2}\right]$$

$$= \frac{1}{\pi iy}.$$

This is the required transform.

Example 13

Using the property of duality, find the Fourier transform of $\frac{1}{x}$.

Solution

We know that $\mathcal{F}\{sgn(x)\} = \frac{1}{\pi iy}$. Therefore by the duality property

$$\mathcal{F}\left\{\frac{1}{x}\right\} = \pi i\, sgn(-y) = -\pi i\, sgn(y).$$

Signum function is an odd function.

Remark: Properties of Fourier transforms

(a) Differentiation rule

If $\frac{df(x)}{dx}$ is absolutely integrable, then

$$\mathcal{F}\left\{\frac{df(x)}{dx}\right\} = (2\pi iy)g(y),$$

where $g(y)$ is the Fourier transform of $f(x)$.

This can be easily proved by considering the inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy = \mathcal{F}^{-1}\{g(y)\}.$$

Hence differentiating both sides with respect to x

$$\frac{df}{dx} = \int_{-\infty}^{\infty} (2\pi iy)g(y)e^{2\pi ixy}dy = \mathcal{F}^{-1}\{(2\pi iy)g(y)\}.$$

Taking Fourier transforms of both sides, we obtain

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = (2\pi iy)g(y)$$
$$= (2\pi iy)\mathcal{F}\{f(x)\}.$$

This result can be extended to the mth derivative such that

$$\mathcal{F}\left\{\frac{d^m f}{dx^m}\right\} = (2\pi iy)^m \mathcal{F}\{f(x)\}. \tag{3.19}$$

(b) Integration rule

The corresponding integration property is

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(z)\,dz\right\} = \frac{1}{2\pi iy}g(y) + g(0)\delta(y).$$

This can be easily established by integrating the inverse Fourier transform,

$$\begin{aligned}\int_{-\infty}^{x} f(z)\,dz &= \int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} g(y)e^{2\pi izy}\,dy\right]dz\\ &= \int_{-\infty}^{\infty} g(y)\left[\int_{-\infty}^{x} e^{2\pi izy}\,dz\right]dy\\ &= \int_{-\infty}^{\infty} g(y)\left[\frac{e^{2\pi ixy}}{2\pi iy} - \lim_{z\to\infty}\frac{e^{-2\pi izy}}{2\pi iy}\right]dy\\ &= \mathcal{F}^{-1}\left\{\frac{g(y)}{2\pi iy}\right\} + \int_{-\infty}^{\infty} g(y)\delta(y).\end{aligned}$$

Therefore, we have

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(z)\,dz\right\} = (2\pi iy)^{-1}g(y) + g(0)\delta(y). \tag{3.20}$$

Example 14

Show that the Fourier transform of x^{-m} is $-\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y)$, where m is an integer and >0.

Proof

We know $\mathcal{F}\{x^{-1}\} = -\pi i\,sgn(y)$ which implies that the inverse transform is $x^{-1} = -\pi i\int_{-\infty}^{\infty} sgn(y)e^{2\pi ixy}\,dy$. Now differentiating both sides $m-1$ times with respect to x and rearranging the terms we obtain

$$\mathcal{F}\{x^{-m}\} = -\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y).$$

Definition 3

x^{-1} is the odd generalized function satisfying $xf(x)=1$; and

$$x^{-m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(x^{-1}). \tag{3.21}$$

Note that using eqn (3.19) of property (a), we can at once find the Fourier transform of x^{-m} from the identity (3.20):

$$\begin{aligned}
\mathcal{F}\{x^{-m}\} &= \frac{(-1)^{m-1}}{(m-1)!}\mathcal{F}\left\{\frac{d^{m-1}}{dx^{m-1}}(x^{-1})\right\} \\
&= \frac{(-1)^{m-1}}{(m-1)!}(2\pi iy)^{m-1}\mathcal{F}\{x^{-1}\} \\
&= \frac{(-1)^{m-1}}{(m-1)!}(2\pi iy)^{m-1}(-\pi i)\, sgn(y) \\
&= -\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\, sgn(y).
\end{aligned}$$

Remark

Integrals involving other inverse-integral power singularities can be interpreted by Definition 3, and this interpretation also has been interpreted by a number of writers. We obtain the following definite integral by using the integration by parts:

$$\begin{aligned}
\int_a^b x^{-m}F(x)\,dx = &\frac{1}{(m-1)!}\int_a^b x^{-1}F^{(m-1)}(x) - \frac{b^{1-m}F(b) - a^{1-m}F(a)}{m-1} \\
&- \frac{b^{2-m}F'(b) - a^{2-m}F'(a)}{(m-1)(m-2)} - \cdots - \frac{b^{-1}F^{(m-2)}(b) - a^{-1}F^{(m-2)}(a)}{(m-1)(m-2)\cdots 1},
\end{aligned}$$

where the Cauchy principal value itself has been left uninterpreted because more readers will be familiar with these integrals. So far we have treated x^n, $x^n\, sgn(x)$ and x^{-m}. It will be seen in the following that more serious difficulties are presented by $x^{-m}\, sgn(x)$. And we will deal with this function as described below.

Example 15

If $f(x) = \frac{d}{dx}(\ln|x|\, sgn(x)) = |x|^{-1}$, then $xf(x) = sgn(x)$. But $f(ax) \neq |a|^{-1}f(x)$.

Proof

$$\begin{aligned} f(x) &= \frac{d}{dx}(\ln|x|\, sgn(x)) \\ &= |x|^{-1}\frac{d|x|}{dx}sgn(x) + \ln|x|\frac{d}{dx}(sgn(x)) \\ &= |x|^{-1}(sgn(x))^2 + \ln|x|(2\delta(x)) \\ &= |x|^{-1}. \end{aligned}$$

Note that for $x \neq 0$, $(sgn(x))^2 = 1$ and $\ln|x|(2\delta(x)) = 0$.

Now it is an easy matter to show that $xf(x) = \frac{x}{|x|} = sgn(x)$.

However, to show the last part we have

$$\begin{aligned} f(ax) &= \frac{d}{adx}(\ln|ax|\, sgn(ax)) \\ &= \frac{d}{adx}[(\ln|x| + \ln|a|)\, sgn(a)\, sgn(x)] \\ &= \frac{1}{|a|}\frac{d}{dx}[\ln|x|\, sgn(x) + \ln|a|\, sgn(x)] \\ &= \frac{1}{|a|}[f(x) + \ln|a|(2\delta(x))]. \end{aligned}$$

In deriving this result we have used that $\frac{d}{dx}(sgn(x)) = 2\delta(x)$. The only satisfactory definition is one which admits the indeterminacy, in the same way as does the definition of the indefinite integral.

Definition 4

The symbol $|x|^{-1}$ will stand for any generalized function $f(x)$ such that $xf(x) = sgn(x)$. The symbol $x^{-m}\, sgn(x)$ will stand for $\frac{(-1)^{m-1}}{(m-1)}$ times the $(m-1)$th derivative of any of the other functions. Thus, $|x|^{-1}$ can be written as

$$\frac{d}{dx}(\ln|x|\, sgn(x)) + C\delta(x) = \frac{d}{dx}\{(\ln|x| + C)\, sgn(x)\}, \tag{3.22}$$

and $x^{-m}\, sgn(x)$ as

$$\frac{(-1)^{m-1}}{(m-1)!}\frac{d^m}{dx^m}\{(\ln|x| + C)\, sgn(x)\} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^m}{dx^m}\{\ln|x|\, sgn(x)\} + C\delta(x), \tag{3.23}$$

where C is any arbitrary constant in each expression (not the same in each). With C being arbitrary, Definition 4 gives $|ax|^{-1} = |a|^{-1}|x|^{-1}$. Again, it is only with C being arbitrary that the relation $x(x^m\, sgn(x)) = x^{-(m-1)}\, sgn(x)$ holds.

Remark

It can be easily proved that

$$\lim_{\varepsilon\to 0}[|x|^{\varepsilon-1} - 2\varepsilon^{-1}\delta(x)] = |x|^{-1}.$$

An equation like this means that the limit exists and equals one of the values of $|x|^{-1}$. It is obtained by differentiating the result

$$\lim_{\varepsilon\to 0}\frac{(|x|^{\varepsilon}-1)\,sgn(x)}{\varepsilon} = \ln|x|\,sgn(x). \tag{3.24}$$

From this relationship, we can obtain the Fourier transform of $\ln|x|$ which is simply $-\frac{1}{2}|y|^{-1}$. The detailed calculations are as follows:

$$\begin{aligned}
\mathcal{F}\{\ln|x|\} &= \mathcal{F}\left\{\lim_{\varepsilon\to 0}\frac{(|x|^{\varepsilon}-1)}{\varepsilon}\right\}\\
&= \lim_{\varepsilon\to 0}[2\cos\pi/2(\varepsilon+1)(\varepsilon)!(2\pi|y|)^{-\varepsilon-1} - \delta(y)]/\varepsilon\\
&= \lim_{\varepsilon\to 0}[-2\sin(\pi/2(\varepsilon))(\varepsilon)!(2\pi|y|)^{-\varepsilon-1} - \delta(y)]/\varepsilon\\
&= -\pi\times(0!)\times(2\pi)^{-1}|y|^{-1} - \lim_{\varepsilon\to 0}\varepsilon^{-1}\delta(y)\\
&= -\frac{1}{2}|y|^{-1}.
\end{aligned}$$

Because for $y \neq 0$, $\lim_{\varepsilon\to 0}\varepsilon^{-1}\delta(y) = 0$. Thus we have that

$$\mathcal{F}\{\ln|x|\} = -\frac{1}{2}|y|^{-1}.$$

Now it is an easy matter to find the Fourier transform of $|x|^{-1}$ by using the duality of Fourier transforms which is simply given by

$$\mathcal{F}\{|x|^{-1}\} = -2(\ln|y| + C),$$

where the arbitrary constant C is present because $|x|^{-1}$ contains an arbitrary multiple of $\delta(x)$.

Example 16

Show that the Fourier transform of $x^{-m}\,sgn(x)$ is $-2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln|y| + C)$.

Proof

We know that $\mathcal{F}\{|x|^{-1}\} = -2(\ln|y| + C)$ and hence we can find the Fourier transform of $x^{-m}\, sgn(x)$ as follows:

$$\begin{aligned}\mathcal{F}\{x^{-m}\, sgn(x)\} &= \frac{(-1)^{m-1}}{(m-1)!}\mathcal{F}\left\{\frac{d^{m-1}}{dx^{m-1}}|x|^{-1}\right\}\\ &= \frac{(-1)^{m-1}}{(m-1)!}(2\pi iy)^{m-1}\mathcal{F}\{|x|^{-1}\}\\ &= \frac{(-1)^{m-1}}{(m-1)!}(2\pi iy)^{m-1}[-2(\ln|y| + C)]\\ &= -2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln|y| + C).\end{aligned}$$

Note that $|x|^{-1} = \frac{d}{dx}[(\ln|x| + C)\, sgn(x)]$.

3.5 Integral powers multiplied by logarithms

3.5.1 The Fourier transform of $x^n \ln|x|$

The generalized function $x^n \ln|x|$ exists and its Fourier transform is given by $-\pi i \frac{n!}{(2\pi iy)^{n+1}}\, sgn(y)$ which can be checked again as follows. We know the Fourier transform of $\ln|x| = -\frac{1}{2}|y|^{-1}$. Using this result we have

$$-\frac{1}{2}|y|^{-1} = \int_{-\infty}^{\infty} \ln|x| e^{-2\pi ixy}\, dx.$$

Differentiating both sides n times with respect to y yields

$$\begin{aligned}-\frac{1}{2}\frac{d^n}{dy^n}(|y|^{-1}) &= (-2\pi i)^n \int_{-\infty}^{\infty} x^n \ln|x| e^{-2\pi ixy}\, dx\\ &= (-2\pi i)^n \mathcal{F}\{x^n \ln|x|\}.\end{aligned}$$

Note that

$$\frac{d}{dy}(|y|^{-1}) = -|y|^{-2}\, sgn(y) = -y^{-2}\, sgn(y).$$

And hence we have

$$\frac{d^n}{dy^n}(|y|^{-1}) = (-1)^n n! y^{-n-1}\, sgn(y).$$

Thus transposing the above result and simplifying we have

$$\mathcal{F}\{x^n \ln|x|\} = -\pi i \frac{n!}{(2\pi iy)^{n+1}} sgn(y). \tag{3.25}$$

3.5.2 The Fourier transform of $x^{-m}\ln|x|$

The generalized function $x^{-m}\ln|x|$ requires a definition, but like x^{-m} itself it presents no difficulty. In fact, if m is even, $|x|^{\alpha}\ln|x|$ tends to a limit as $\alpha \to -m$.

$$\begin{aligned}
\mathcal{F}\{|x|^{\alpha}\ln|x|\} &= \mathcal{F}\left\{\frac{\partial}{\partial\alpha}|x|^{\alpha}\right\} \\
&= \frac{\partial}{\partial\alpha}\mathcal{F}\{|x|^{\alpha}\} \\
&= \frac{\partial}{\partial\alpha}\left\{(2\cos\pi/2(\alpha+1))\alpha!(2\pi|y|)^{-\alpha-1}\right\} \\
&= \Big[2\cos\pi/2(\alpha+1)\alpha!(2\pi|y|)^{-\alpha-1} \\
&\qquad \times\left(-\ln(2\pi|y|)+\psi(\alpha)-\frac{\pi}{2}\tan\frac{\pi}{2}(\alpha+1)\right)\Big],
\end{aligned}$$

where $\psi(\alpha) = \frac{d}{d\alpha}(\ln\alpha!)$. Now using the standard formula $\alpha!(-\alpha-1)! = -\pi\operatorname{cosec}\pi\alpha$ and its logarithmic derivative we can set the above transform as

$$\begin{aligned}
\mathcal{F}\{|x|^{\alpha}\ln|x|\} &= \frac{\pi}{(-\alpha-1)!\cos\frac{1}{2}\pi\alpha}(2\pi|y|)^{-\alpha-1} \\
&\quad\times\left[-\ln(2\pi|y|)+\psi(-\alpha-1)-\pi\cot\pi\alpha+\frac{1}{2}\cot\frac{1}{2}\pi\alpha\right],
\end{aligned} \tag{3.26}$$

and this tends to a limit

$$\mathcal{F}\{|x|^{-m}\ln|x|\} = \pi i\frac{(-2\pi i y)^{m-1}}{(m-1)!}sgn(y)[\ln(2\pi|y|)-\psi(m-1)] \tag{3.27}$$

as $\alpha \to -m$.

Similarly, we may find that $|x|^{\alpha}\ln|x|\,sgn(x)$ tends to a limit $\alpha \to -m$ if m is odd, from the fact that the Fourier transform (3.16) tends to a limit, which again can be thrown into the form (3.27). These facts make the following definition appropriate.

Definition 5

The generalized function $x^{-m}\ln|x|$ is the function whose Fourier transform is eqn (3.27), so that it is the limit as $\alpha \to -m$ of $|x|^{\alpha}\ln|x|$ if m is even and of $|x|^{\alpha}\ln|x|\,sgn(x)$ if m is odd.

Fourier inversion theorem applied to Definition 5 gives us that

$$\begin{aligned}
\mathcal{F}\{x^{n}\ln|x|\,sgn(x)\} &= \int_{-\infty}^{\infty}(x^{n}\ln|x|\,sgn(x))e^{-2\pi ixy}\,dx \\
&= -2\frac{n!}{(2\pi iy)^{n+1}}[\ln(2\pi|y|)-\psi(n)].
\end{aligned} \tag{3.28}$$

In connection with eqns (3.27) and (3.28) the reader may like to be reminded that

$$\psi(n) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \tag{3.29}$$

where $\gamma = -\psi(0) = 0.5772$ is Euler's constant.

3.5.3 The Fourier transform of $x^{-m}\ln|x|\,sgn(x)$

Finally, we come to the generalized function $x^{-m}\ln|x|\,sgn(x)$, which like $x^{-m}\,sgn(x)$ presents more serious difficulties, and involves a certain indeterminacy. It is most expeditiously approached from the special case $n=0$ of eqn (3.28); the Fourier transform of $\ln|x|\,sgn(x)$ is

$$\begin{aligned}\mathcal{F}\{\ln|x|\,sgn(x)\} &= -\frac{1}{\pi i y}(\ln(2\pi|y|)+\gamma)\\ &= -\frac{1}{2\pi i}\frac{d}{dy}(\ln(2\pi|y|)+\gamma)^2.\end{aligned} \tag{3.30}$$

If now $f(x)$ is taken as the function whose Fourier transform is $(\ln(2\pi|y|)+\gamma)^2$, it follows from eqn (3.30) and $\mathcal{F}\{x^n f(x)\} = (-2\pi i)^{-n} g^{(n)}(y)$ that

$$xf(x) = \ln|x|\,sgn(x). \tag{3.31}$$

However, we know that *if $g(y)$ is a generalized function and $yg(y)=0$, then $g(y)$ is a constant times $\delta(y)$*, the general solution of eqn (3.31) is $f(x)+C\delta(x)$, there is no way of selecting one solution as more suitable than the other and, indeed, if C is not left arbitrary, the ordinary rules of manipulation cannot be applied to this function and those derived from it by differentiation.

Definition 6

The symbol $|x|^{-1}\ln|x|$ will stand for any generalized function $f(x)$ which satisfies eqn (3.31). The symbol $x^{-m}\ln|x|\,sgn(x)$ will stand for

$$\frac{(-1)^{m-1}}{(m-1)!}f^{(m-1)}(x) + \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-n}\right]x^{-m}\,sgn(x), \tag{3.32}$$

where $f(x)$ is any generalized function which satisfies eqn (3.31).

Remark

Equation (3.32) is the equation relating the corresponding ordinary function. Under Definition 6, the Fourier transform of $|x|^{-1}\ln|x|$ is

$$\mathcal{F}\{|x|^{-1}\ln|x|\} = (\ln(2\pi|y|)+\gamma)^2 + C, \tag{3.33}$$

where C is arbitrary, and then the Fourier transform of $x^{-m}\ln|x|\,sgn(x)$ can be written at once into the form

$$\mathcal{F}\{x^{-m}\ln|x|\,sgn(x)\} = \frac{(-2\pi iy)^{m-1}}{(m-1)!}[\{\ln(2\pi|y|) - \psi(m-1)\}^2 + C], \qquad (3.34)$$

where C is again arbitrary (and not in general the same in each formula).

3.6 Summary of results of Fourier transforms

This section deals with the summary of the complete set of Fourier transform formulae, for the elementary functions possessing the algebraic or algebraico-logarithmic singularities at the origin $x = 0$. These formulations have been derived in the previous sections, and here we shall formally collect them in a systematic way. These formulae will be of immense use in the study of the asymptotic behaviour of the Fourier transform of a function as $|y| \to \infty$ by inspecting its singularities which will be discussed in the next chapter. We shall now present these formulae one by one sequentially highlighting the different steps leading to the result.

Formula 1: $\mathcal{F}\{|x|^\alpha\}$

$$\begin{aligned}
\mathcal{F}\{|x|^\alpha\} &= \mathcal{F}\{x^\alpha H(x) + (-x)^\alpha H(-x)\} \\
&= \alpha!(2\pi|y|)^{-\alpha-1}\{e^{\frac{\pi}{2}i(\alpha+1)} + e^{-\frac{\pi}{2}i(\alpha+1)}\} \\
&= 2\cos\left(\frac{\pi}{2}(\alpha+1)\right)\alpha!(2\pi|y|)^{-\alpha-1} \\
&= -2\sin\left(\frac{\pi}{2}\alpha\right)\alpha!(2\pi|y|)^{-\alpha-1}.
\end{aligned}$$

Formula 2: $\mathcal{F}\{|x|^\alpha\, sgn(x)\}$

$$\begin{aligned}
\mathcal{F}\{|x|^\alpha\, sgn(x)\} &= \int_0^\infty x^\alpha e^{-2\pi ixy}\,dx - \int_{-\infty}^0 (-x)^\alpha e^{-2\pi ixy}\,dx \\
&= \alpha!(2\pi|y|)^{-\alpha-1}\{e^{-\frac{\pi}{2}i(\alpha+1)} - e^{\frac{\pi}{2}i(\alpha+1)}\}\,sgn(y) \\
&= -2i\sin\left(\frac{\pi}{2}(\alpha+1)\right)\alpha!(2\pi|y|)^{-\alpha-1}\,sgn(y) \\
&= -2i\cos\left(\frac{\pi}{2}\alpha\right)\alpha!(2\pi|y|)^{-\alpha-1}\,sgn(y).
\end{aligned}$$

Formula 3: $\mathcal{F}\{|x|^{\alpha}H(x)\}$

$$\begin{aligned}\mathcal{F}\{|x|^{\alpha}H(x)\} &= \int_0^{\infty} x^{\alpha}e^{-2\pi ixy}\,dx \\ &= \alpha!(2\pi iy)^{-\alpha-1} \\ &= \alpha!(2\pi|y|)^{-\alpha-1}(e^{-\frac{1}{2}i(\alpha+1)\,sgn(y)}).\end{aligned}$$

We can use the relation between $H(x)$ and $sgn(x)$ that means $H(x) = \frac{1}{2}[1 + sgn(x)]$.

$$\begin{aligned}\mathcal{F}\{|x|^{\alpha}H(x)\} &= \frac{1}{2}[\mathcal{F}\{|x|^{\alpha}(1 + sgn(x))\} \\ &= \alpha!(2\pi|y|)^{-\alpha-1}\{e^{-\frac{\pi}{2}i(\alpha+1)\,sgn(y)}\}.\end{aligned}$$

Formula 4: $\mathcal{F}\{x^n\}$

We know that

$$\delta(y) = \mathcal{F}\{1\} = \int_{-\infty}^{\infty}(1)e^{-2\pi ixy}\,dx.$$

Differentiating n times both sides with respect to y and transposing the result yields

$$\mathcal{F}\{x^n\} = (-2\pi i)^{-n}\delta^{(y)}.$$

Formula 5: $\mathcal{F}\{x^n\,sgn(x)\}$

We know that

$$\frac{2}{2\pi iy} = \mathcal{F}\{sgn(x)\} = \int_{-\infty}^{\infty} sgn(x)e^{-2\pi ixy}\,dx.$$

Now differentiating both sides n times with respect to y and transposing the result yields

$$\mathcal{F}\{x^n\,sgn(x)\} = 2n!(2\pi iy)^{-n-1}.$$

Formula 6: $\mathcal{F}\{x^nH(x)\}$

$$\begin{aligned}\mathcal{F}\{x^nH(x)\} &= \left[\frac{1}{2}\mathcal{F}\{x^n\} + \frac{1}{2}\mathcal{F}\{x^n sgn(x)\}\right] \\ &= (-2\pi i)^{-n}\frac{1}{2}\delta^{(y)} + n!(2\pi iy)^{-n-1} \\ &= (-2\pi i)^{-n}\left\{\frac{1}{2}\delta^{(y)} + \frac{(-1)^n n!}{2\pi iy^{n+1}}\right\}.\end{aligned}$$

Formula 7: $\mathcal{F}\{x^{-m}\}$

We know that $\mathcal{F}\{x^{-1}\} = -\pi i\, sgn(y)$. The inverse is $x^{-1} = \int_{-\infty}^{\infty} (-\pi i\, sgn(y))e^{2\pi ixy}\, dy$. And also we know that $x^{-m} = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}}(x^{-1})$. Hence we can easily obtain the result.

$$\begin{aligned}
\mathcal{F}\{x^{-m}\} &= \frac{(-1)^{m-1}}{(m-1)!}\mathcal{F}\left\{\frac{d^{m-1}}{dx^{m-1}}(x^{-1})\right\} \\
&= \frac{(-1)^{m-1}}{(m-1)!}(-\pi i)(2\pi iy)^{m-1}\, sgn(y) \\
&= -\pi i \frac{(-2\pi iy)^{m-1}}{(m-1)!} sgn(y).
\end{aligned}$$

Formula 8: $\mathcal{F}\{x^{-m}\, sgn(x)\}$

We know $\mathcal{F}\{|x|^{-1}\} = -(2\ln|y| + C)$.

$$\begin{aligned}
\mathcal{F}\{x^{-m}\, sgn(x)\} &= \frac{(-1)^{m-1}}{(m-1)!}\mathcal{F}\left\{\frac{d^{m-1}}{dx^{m-1}}(|x|^{-1})\right\} \\
&= -2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln|y| + C).
\end{aligned}$$

Formula 9: $\mathcal{F}\{x^{-m}H(x)\}$

$$\begin{aligned}
\mathcal{F}\{x^{-m}H(x)\} &= \mathcal{F}\left\{x^{-m}\left[\frac{1}{2} + \frac{1}{2}\, sgn(x)\right]\right\} \\
&= -\frac{(-2\pi iy)^{m-1}}{(m-1)!}\left[\frac{1}{2}\pi i\, sgn(y) + \ln|y| + C\right].
\end{aligned}$$

Formula 10: $\mathcal{F}\{|x|^{\alpha} \ln|x|\}$

$$\begin{aligned}
\mathcal{F}\{|x|^{\alpha}\ln|x|\} &= \left(2\cos\frac{1}{2}\pi(\alpha+1)\right)\alpha!(2\pi|y|)^{-\alpha-1} \\
&\quad\times\left(-2\ln(2\pi|y|) + \psi(\alpha) - \frac{1}{2}\tan\frac{1}{2}(\alpha+1)\right).
\end{aligned}$$

Formula 11: $\mathcal{F}\{|x|^{\alpha}\ln|x|\,sgn(x)\}$

$$\mathcal{F}\{|x|^{\alpha}\ln|x|\,sgn(x)\} = \left(-2i\,\sin\frac{1}{2}\pi(\alpha+1)\right)\alpha!(2\pi|y|)^{-\alpha-1}\,sgn(y)$$
$$\times\left(-2\ln(2\pi|y|)+\psi(\alpha)+\frac{1}{2}\cot\frac{1}{2}(\alpha+1)\right).$$

Formula 12: $\mathcal{F}\{|x|^{\alpha}\ln|x|H(x)\}$

$$\mathcal{F}\{|x|^{\alpha}\ln|x|H(x)\} = (e^{-\frac{1}{2}\pi i(\alpha+1)\,sgn(y)})\alpha!(2\pi|y|)^{-\alpha-1}$$
$$\times\left\{-\ln(2\pi|y|)+\psi(\alpha)-\frac{1}{2}\pi i\,sgn(y)\right\}.$$

Formula 13: $\mathcal{F}\{x^{n}\ln|x|\}$

We know that $\mathcal{F}\{|x|^{-1}\} = -2\ln|y|$. Hence by duality theorem $\mathcal{F}\{\ln|x|\} = -\frac{1}{2}|y|^{-1}$. With this information we can obtain

$$\mathcal{F}\{x^{n}\ln|x|\} = -\pi i\frac{n!}{(2\pi iy)^{n+1}}sgn(y).$$

Formula 14: $\mathcal{F}\{x^{n}\ln|x|\,sgn(x)\}$

We know from Formula 5 that $\mathcal{F}\{x^{n}\,sgn(x)\} = 2(n!)(2\pi iy)^{-n-1}$. Hence we have

$$\mathcal{F}\{x^{n}\ln|x|\,sgn(x)\} = \frac{d}{dn}[2(n!)(2\pi iy)^{-n-1}]$$
$$= -2\frac{n!}{(2\pi iy)^{n+1}}[\ln(2\pi|y|)-\psi(n)],$$

where $\psi(n) = \frac{d\ln(n!)}{dn} = \frac{1}{n!}\frac{d(n!)}{dn} = \frac{1}{n}$.

Formula 15: $\mathcal{F}\{x^{n}\ln|x|H(x)\}$

We know that $H(x) = \frac{1}{2}+\frac{1}{2}\,sgn(x)$. Hence using this relation yields

$$\mathcal{F}\{x^{n}\ln|x|H(x)\} = \mathcal{F}\left\{x^{n}\ln|x|\left(\frac{1}{2}+\frac{1}{2}\,sgn(x)\right)\right\}$$
$$= \frac{1}{2}\left[-\pi i\frac{n!}{(2\pi iy)^{n+1}}\,sgn(y)-2\frac{n!}{(2\pi iy)^{n+1}}[\ln(2\pi|y|)-\psi(n)]\right]$$
$$= -\frac{n!}{(2\pi iy)^{n+1}}\times\left[\frac{1}{2}\pi i\,sgn(y)+\ln(2\pi|y|)-\psi(n)\right].$$

Formula 16: $\mathcal{F}\{x^{-m}\ln|x|\}$

From Formula 7, we know $\mathcal{F}\{x^{-m}\} = -\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y)$. Hence it is an easy matter to find the required transform:

$$\begin{aligned}\mathcal{F}\{x^{-m}\ln|x|\} &= \frac{d}{dm}[\mathcal{F}\{x^{-m}\}]\\ &= \frac{d}{dm}\left[-\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y)\right]\\ &= \pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y)\times[\ln(2\pi|y|)-\psi(m-1)].\end{aligned}$$

Formula 17: $\mathcal{F}\{x^{-m}\ln|x|\,sgn(x)\}$

Formula 8 tells us $\mathcal{F}\{x^{-m}\,sgn(x)\} = -2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln|y|+C)$. Hence we have

$$\begin{aligned}\mathcal{F}\{x^{-m}\ln|x|\,sgn(x)\} &= \frac{d}{dm}[\mathcal{F}\{x^{-m}\,sgn(x)\}]\\ &= \frac{d}{dm}\left[-2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln|y|+C)\right]\\ &= \frac{(-2\pi iy)^{m-1}}{(m-1)!}\times[(\ln(2\pi|y|)-\psi(m-1))^2+C].\end{aligned}$$

Formula 18: $\mathcal{F}\{x^{-m}\ln|x|H(x)\}$

$$\begin{aligned}\mathcal{F}\{x^{-m}\ln|x|H(x)\} &= \mathcal{F}\left\{x^{-m}\ln|x|\left(\frac{1}{2}+\frac{1}{2}\,sgn(x)\right)\right\}\\ &= \frac{1}{2}\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\,sgn(y)\times[\ln(2\pi|y|)-\psi(m-1)]\\ &\quad+\frac{1}{2}\frac{(-2\pi iy)^{m-1}}{(m-1)!}\times[(\ln(2\pi|y|)-\psi(m-1))^2+C]\\ &= \frac{(-2\pi iy)^{m-1}}{(m-1)!}\left[\frac{1}{2}\left(\frac{1}{2}\pi i\,sgn(y)+\ln(2\pi|y|)-\psi(m-1)\right)^2+C\right].\end{aligned}$$

As in many applications, the singularities as cited in the foregoing Formulae 1–18 are not always at the origin, the formula which expresses the Fourier transform of $f(ax+b)$ in terms of that of $f(x)$ must frequently be used in conjunction with Formulae 1–18. In fact, if $\mathcal{F}\{f(x)\} = g(y)$, then $\mathcal{F}\{f(ax+b)\} = \frac{1}{|a|}e^{2\pi iby/a}g(\frac{y}{a})$. Conversely, $\mathcal{F}\{e^{ikx}f(x)\} = g(y-\frac{k}{2\pi})$.

In many applications, terms involving higher powers of $\ln|x|$ are present. The process to derive the Fourier transform of a function involving $(\ln|x|)^2$ or higher powers by the method described in this chapter is as follows. For example, suppose we need to determine the Fourier transform of the function $|x|^\alpha(\ln|x|)^2$. This can be at once achieved by

$$\mathcal{F}\{|x|^\alpha(\ln|x|)^2\} = \frac{\partial}{\partial\alpha}\mathcal{F}\{|x|^\alpha(\ln|x|)\} = \frac{\partial^2}{\partial\alpha^2}\mathcal{F}\{|x|^\alpha\}.$$

This result can be extended to higher powers, say nth power.

$$\begin{aligned}\mathcal{F}\{|x|^\alpha(\ln|x|)^n\} &= \frac{\partial}{\partial\alpha}\mathcal{F}\{|x|^\alpha(\ln|x|)^{n-1}\}\\ &\cdots\\ &= \frac{\partial^n}{\partial\alpha^n}\mathcal{F}\{|x|^\alpha\}.\end{aligned}$$

Thus using Formulae 1–18, the Fourier transform can be easily determined.

It is worth noting that the Fourier transforms of any rational functions can be obtained by using these formulae. By a familiar theorem in algebra any rational function can be expressed "in partial fraction" or more precisely as a linear combination of the integral power of x^n and negative integral powers of $(x-c)^{-m}$ for different real and complex values of c. The Fourier transform of x^n, and of $(x-c)^{-m}$ for real c, can be easily obtained from these formulae. For complex c we need the following additional result.

Example 17

Show that the Fourier transform of $\{(x-(c_1+ic_2))\}^{-m}$ for $c_2 \neq 0$ is

$$2\pi i H(-c_2 y)\, sgn(c_2)\frac{(-2\pi i y)^{m-1}}{(m-1)!}e^{-2\pi i y(c_1+ic_2)}. \tag{3.35}$$

Proof

Let us consider that

$$f(x) = (2\pi i)^m \frac{x^{m-1}}{(m-1)!}e^{-2\pi|c_2|x}H(x). \tag{3.36}$$

Hence

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{(2\pi i)^m}{(m-1)!}\int_0^\infty (x^{m-1}e^{-2\pi|c_2|x})e^{-2\pi i x y}\,dx\\ &= \frac{(2\pi i)^m}{(m-1)!}\int_0^\infty x^{m-1}e^{-2\pi i(y-i|c_2|)x}\,dx\end{aligned}$$

$$= (y - i|c_2|)^{-m}$$
$$= g(y).$$

By using the theorem of duality we have

$$\mathcal{F}\{(x - ic_2)^{-m}\} = (2\pi i)^m \frac{(-y)^{m-1}}{(m-1)!} e^{-2\pi|c_2|(-y)} H(-yc_2)$$
$$= (2\pi i)\, sgn(c_2) \frac{(-2\pi iy)^{m-1}}{(m-1)!} e^{2\pi c_2 y} H(-yc_2). \qquad (3.37)$$

Hence we have

$$\mathcal{F}\{(x - (c_1 + ic_2))^{-m}\} = 2\pi i H(-c_2 y)\, sgn(c_2) \frac{(-2\pi iy)^{m-1}}{(m-1)!} e^{-2\pi iy(c_1+ic_2)}.$$

Note that in obtaining this result we have used the theorem that if $\mathcal{F}\{f(x)\} = g(y)$ then $\mathcal{F}\{f(x - c_1)\} = e^{-2\pi ic_1 y} g(y)$.

Remark

It is worth noting that to obtain eqn (3.35) as the Fourier transform of $\{x - (c_1 + ic_2)\}$ we must first derive the expression for $f(x)$ in eqn (3.36) from our previous result. This is the main starting point in evaluating the Fourier transform of this type of functions with a complex argument. The duality of the transform plays an important role in getting the result. The frequency shifting theorem stated above is also a key factor. It is to be noted that the Fourier transform of $\{x - c\}$ is not the limit of that of $\{x - (c_1 + ic_2)\}$ as $c_2 \to 0$ either from above or below, but that (since $H(-c_2 y) + H(c_2 y) = 1$) it is half the sum of the two limits. Clear manifestation of this point can be attributed to a contour-integration approach to the evaluation of Fourier integrals. A little note on the Fourier integral

$$\mathcal{F}\{(x - c)^{-m}\} = \int_{-\infty}^{\infty} (x - c)^{-m} e^{-2\pi ixy}\, dx.$$

Here $c = c_1 + ic_2$ is a complex number. This integral has multiple pole at $x = c$ of order m. Hence by using a semi-circular contour of infinite radius, we see that the contribution to the integral comes from the residue at the pole $x = c$, and so the value of the integral is equal to $(2\pi i) \times$ the residue.

$$\mathcal{F}\{(x - c)^{-m}\} = \int_{-\infty}^{\infty} (x - c)^{-m} e^{-2\pi ixy}\, dx$$
$$= (2\pi i) \times \frac{(-2\pi iy)}{(m-1)!} e^{-2\pi iy(c_1+ic_2)}.$$

We conclude this chapter demonstrating the following examples for the benefit of the reader.

Example 18

Show that $\int_{-\infty}^{\infty} x^{-4}e^{-x^2}\,dx = \frac{4}{3}\sqrt{\pi}$.

Proof

The proof can be effected by integration by parts and then using the standard result $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

$$\begin{aligned}\int_{-\infty}^{\infty} x^{-4}e^{-x^2}\,dx &= -\frac{2}{3}\int_{-\infty}^{\infty} x^{-2}e^{-x^2}\,dx\\ &= \frac{4}{3}\int_{-\infty}^{\infty} e^{-x^2}\,dx\\ &= \frac{8}{3}\int_0^{\infty} e^{-x^2}\,dx\\ &= \frac{4}{3}\sqrt{\pi}.\end{aligned}$$

Hence this is the required result.

Example 19

Prove that the Fourier transform of $f(x) = \frac{x^5}{x^4-1}$ is $g(y) = \frac{i\delta'(y)}{2\pi} + \frac{1}{2}\pi i(e^{-2\pi|y|} - \cosh 2\pi y)\, sgn(y)$.

Proof

In partial fractions, $f(x)$ can be rewritten as

$$f(x) = x + \frac{1}{4}\left\{\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x+i} - \frac{1}{x-i}\right\}.$$

Now taking the Fourier transforms of both sides we obtain

$$g(y) = \mathcal{F}\{x\} + \frac{1}{4}\mathcal{F}\left\{\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x+i} - \frac{1}{x-i}\right\}.$$

We know $\mathcal{F}\{1\} = \delta(y)$. And so

$$\delta(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy}\, dx.$$

Differentiating with respect to y we have

$$\begin{aligned}\delta'(y) &= (-2\pi i)\int_{-\infty}^{\infty} x^{-2\pi ixy}\, dx \\ &= (-2\pi i)\mathcal{F}\{x\}.\end{aligned}$$

And hence, transposing the terms and simplifying we obtain

$$\mathcal{F}\{x\} = \frac{i\delta'(y)}{2\pi}.$$

Next we calculate the Fourier transform of $\frac{1}{x+1}$.

$$\mathcal{F}\left\{\frac{1}{x+1}\right\} = \int_{-\infty}^{\infty} \frac{1}{x+1} e^{-2\pi ixy}\, dx$$

Changing $x+1$ to x on the right-hand side, we have

$$\begin{aligned}&= e^{2\pi iy}\int_{-\infty}^{\infty} \frac{1}{x} e^{-2\pi ixy}\, dx \\ &= e^{2\pi iy}\mathcal{F}\left\{\frac{1}{x}\right\} \\ &= e^{2\pi iy}(-\pi i)\, sgn(y).\end{aligned}$$

Similarly, we have

$$\mathcal{F}\left\{\frac{1}{x-1}\right\} = \int_{-\infty}^{\infty} \frac{1}{x+1} e^{-2\pi ixy}\, dx$$

Changing $x-1$ to x on the right-hand side, we have

$$\begin{aligned}&= e^{-2\pi iy}\int_{-\infty}^{\infty} \frac{1}{x} e^{-2\pi ixy}\, dx \\ &= e^{-2\pi iy}\mathcal{F}\left\{\frac{1}{x}\right\} \\ &= e^{-2\pi iy}(-\pi i)\, sgn(y).\end{aligned}$$

Adding these two results we have

$$\mathcal{F}\left\{\frac{1}{x+1} + \frac{1}{x-1}\right\} = \cos 2\pi y(-2\pi i\, sgn(y)).$$

In the same way, we can write at once

$$\mathcal{F}\left\{\frac{1}{x+i}\right\} = e^{-2\pi y}(-\pi i\, sgn(y)),$$
$$\mathcal{F}\left\{\frac{1}{x-i}\right\} = e^{2\pi y}(-\pi i\, sgn(y)).$$

Adding these two results yields

$$\mathcal{F}\left\{\frac{1}{x+i}+\frac{1}{x-i}\right\} = \cosh 2\pi y(-2\pi i\, sgn(y)).$$

Thus, gathering all these information we can write the answer as follows:

$$\mathcal{F}\left\{\frac{x^5}{x^4-1}\right\} = \frac{i\delta'(y)}{2\pi} + \frac{1}{2}\{\cos 2\pi y - \cosh 2\pi y\}(-\pi i\, sgn(y)).$$

Hence this is the required result.

Example 20

Find the Fourier transform of $(x^2+5x+4)^{-2}$.

Solution

The given algebraic expression can be written by partial fraction as follows:

$$(x^2+5x+4)^{-2} = \frac{1}{9}\left[\frac{1}{(x+1)^2}+\frac{1}{(x+4)^2}\right] - \frac{2}{27}\left[\frac{1}{(x+1)}-\frac{1}{(x+4)}\right].$$

We know that $\mathcal{F}\{\frac{1}{x}\} = (-\pi i)\, sgn(y)$ and also $\mathcal{F}\{\frac{1}{x^2}\} = (-2\pi i y)(-\pi i)\, sgn(y) = (-2\pi^2 y)\, sgn(y)$. We shall use these formulae to obtain the Fourier transform of the given function.

$$\begin{aligned}
\mathcal{F}\{(x^2+5x+4)^{-2}\} &= \mathcal{F}\left\{\frac{1}{9}\left[\frac{1}{(x+1)^2}+\frac{1}{(x+4)^2}\right] - \frac{2}{27}\left[\frac{1}{(x+1)}-\frac{1}{(x+4)}\right]\right\} \\
&= \frac{1}{9}[e^{2\pi iy}(-2\pi^2 y\, sgn(y)) + e^{8\pi iy}(-2\pi^2 y)\, sgn(y)] \\
&\quad - \frac{2}{27}[e^{2\pi iy}(-\pi i)\, sgn(y) - e^{8\pi iy}(-\pi i)\, sgn(y)] \\
&= \frac{1}{9}\left\{\left(\frac{2\pi i}{3} - 2\pi^2 y\right)e^{2\pi iy} - \left(\frac{2\pi i}{3} + 2\pi^2 y\right)e^{8\pi iy}\right\} sgn(y).
\end{aligned}$$

This is the required result.

Example 21

Find the Fourier transform of $(x^2+2x+5)^{-2}$.

Solution

Using partial fractions, $(x^2+2x+5)^{-2}$ can be expressed as

$$-\frac{1}{16}\left\{\frac{1}{(x+a)^2}+\frac{1}{(x+a^*)^2}+(8i)\left[\frac{1}{x+a}-\frac{1}{x+a^*}\right]\right\},$$

where $a=1+2i$ and the complex conjugate of a is $a^*=1-2i$. Using the same procedure as above, we obtain

$$\mathcal{F}\left\{\frac{1}{(x+a)^2}\right\}=e^{2\pi iya}(-2\pi^2 y)\,sgn(y)=e^{-4\pi y+2\pi iy}(-2\pi^2 y)\,sgn(y),$$

$$\mathcal{F}\left\{\frac{1}{(x+a^*)^2}\right\}=e^{2\pi iya^*}(-2\pi^2 y)\,sgn(y)=e^{4\pi y+2\pi iy}(-2\pi^2 y)\,sgn(y),$$

$$\mathcal{F}\left\{\frac{1}{(x+a)}\right\}=e^{2\pi iya}(-\pi iy)\,sgn(y)=e^{-4\pi y+2\pi iy}(-\pi iy)\,sgn(y),$$

$$\mathcal{F}\left\{\frac{1}{(x+a^*)}\right\}=e^{2\pi iya^*}(-\pi iy)\,sgn(y)=e^{4\pi y+2\pi iy}(-\pi iy)\,sgn(y).$$

Now collecting all these information we can at once write the result as follows:

$$\begin{aligned}\mathcal{F}\{(x^2+2x+5)^{-2}\} &= -\frac{1}{16}[(e^{-4\pi y+2\pi iy}(-2\pi^2 y)\,sgn(y)\\ &\quad + e^{4\pi y+2\pi iy}(-2\pi^2 y)\,sgn(y))+8i(e^{-4\pi y+2\pi iy}(-\pi iy)\,sgn(y)\\ &\quad - e^{4\pi y+2\pi iy}(-\pi iy)\,sgn(y))]\\ &= \frac{1}{4}[\pi y\ \cosh 4\pi y+4\ \sinh 4\pi y](\pi e^{2\pi iy})\,sgn(y).\end{aligned}$$

Example 22

Find the Fourier transform of $(1-x)^{-3/2}\ln(1-x)H(1-x)$.

Solution

Let us consider that $-\frac{3}{2} = \alpha$. Hence we have the function as $(1-x)^{\alpha}\ln(1-x)$ $H(1-x)$. Thus

$$\begin{aligned}
\mathcal{F}\{(1-x)^{\alpha}\ln(1-x)H(1-x)\} &= \int_{-\infty}^{\infty}[(1-x)^{\alpha}\ln(1-x)H(1-x)]e^{-2\pi ixy}\,dx \\
&= \int_{-\infty}^{1}(1-x)^{\alpha}\ln(1-x)e^{-2\pi ixy}\,dx \\
&\text{Substituting } 1-x=z \text{ we obtain} \\
&= e^{-2\pi iy}\int_{0}^{\infty}z^{\alpha}\ln|z|\,e^{2\pi izy}\,dz \\
&= e^{-2\pi iy}\frac{\partial}{\partial\alpha}\int_{0}^{\infty}z^{\alpha}\,e^{2\pi izy}\,dz \\
&= e^{-2\pi iy}\frac{\partial}{\partial\alpha}[\alpha!(-2\pi iy)^{-\alpha-1}] \\
&= e^{-2\pi iy}\alpha![\psi(\alpha)(-2\pi iy)^{-\alpha-1} \\
&\quad+(-2\pi iy)^{-\alpha-1}\ln(|-2\pi iy)|] \\
&= e^{-2\pi iy}\alpha![\psi(\alpha)(-2\pi iy)^{-\alpha-1} \\
&\quad+(-2\pi iy)^{-\alpha-1}\ln(2\pi|y|)] \\
&= e^{-2\pi iy}\alpha!(-2\pi iy)^{-\alpha-1}[\psi(\alpha)+\ln(2\pi|y|)],
\end{aligned}$$

where $\psi(\alpha) = \frac{d}{d\alpha}(\ln\alpha!)$. Note that $\alpha!(-\alpha-1)! = \Gamma(\alpha+1)\Gamma(-\alpha) = -\pi\operatorname{cosec}\pi\alpha$. And also $(-\frac{3}{2})! = -2\sqrt{\pi}$ because $(\frac{1}{2})! = \frac{1}{2}\sqrt{\pi}$.

3.7 Exercises

1. Evaluate the following integrals.
   (a) $\int_{-\infty}^{\infty}\exp(-x^2)\,\delta(x-a)\,dx$
   (b) $\int_{-\infty}^{\infty}\exp(-x^2)\,H(x)\,dx$
   (c) $\int_{-\infty}^{\infty}\exp(-x^2)\,sgn(x)\,dx$
   (d) $\int_{-\infty}^{\infty}\operatorname{sech}x\,H(x-a)\,dx$.

2. Prove that

$$x^n\delta^{(m)}(x) = \begin{Bmatrix}(-1)^n\frac{m!}{(m-n)!}\delta^{(m-n)}(x) & m\geq n \\ 0 & m<n\end{Bmatrix}.$$

3. Evaluate the following integrals.

   (a) $\int_{-\infty}^{\infty} \exp(-x^2)\ \delta'(x-a)\, dx$

   (b) $\int_{-\infty}^{\infty} \operatorname{sech} x\ \delta'(x-a)\, dx$

   (c) $\int_{-\infty}^{\infty} \exp(-x^2)\ H'(x-a)\, dx$

   (d) $\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\, dx = (-1)^n f^{(n)}(o)$.

4. Prove that $|x|' = sgn(x)$.
5. Find the Fourier transform of $x^m\ \delta^{(n)}(x)$, m and n being positive integers.
6. If the Fourier transform of $f(x)$ is $g(y)$, then find the Fourier transform of $x^n f(x)$, n being a positive integer.
7. Verify the following:

   (a) $\lim_{\alpha\to 0} \frac{1}{\alpha}(|x|^{\alpha} - 1)\, sgn(x) = sgn(x)$

   (b) $\lim_{\lambda\to\infty} \frac{\sin \lambda x}{\pi x} = \delta(x)$

   (c) $\lim_{\lambda\to\infty} \frac{2}{\pi} \arctan(\lambda x) = sgn(x)$.

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4 Asymptotic estimation of Fourier transforms

4.1 Introduction

We define an asymptotic expression as follows. In accordance with Lighthill's concept, an asymptotic expression for a function is an expression as the sum of a simpler function and of a remainder which tends to zero at infinity, or (more generally) which tends to zero after multiplication by some power. It is worth noting here that it is not always possible to know the Fourier transforms of certain functions. And in that situation, when we do not know the Fourier transform of a given function, it is usually advisable for convenience to possess at least an asymptotic expression for it so that some physical meaning can be derived. This chapter deals with the development of a method which leads quickly to such an asymptotic expression for most functions occurring in applications (Jones, 1982; Rahman, 2001; Temple, 1953; 1955).

4.2 The Riemann–Lebesgue lemma

The Riemann–Lebesgue method involves writing the given function, say $f(x)$, as the sum of a simpler function $F(x)$, whose Fourier transform $G(y)$ we know, and of a remainder $f_R(x)$, whose Fourier transform $g_R(y)$ tends to zero, or (more generally) is such that the Fourier transform $(2\pi iy)^N g_R(y)$ of its nth derivative $f_R^{(N)}(x)$ tends to zero. Then the Fourier transform of $f(x)$, say $g(y)$, satisfies

$$g(y) = G(y) + g_R(y) = G(y) + o(|y|^{-N}) \tag{4.1}$$

as $|y| \to \infty$.

To develop such a method, we need a simple technique for recognizing functions whose Fourier transforms must tend to zero as $|y| \to \infty$. The following theorem is the classical result for ordinary functions.

Theorem 4.1: The Riemann–Lebesgue lemma

If $f(x)$ is an ordinary function absolutely integrable from $-\infty$ to ∞, and $g(y)$ is the Fourier transform, then $g(y) \to 0$ as $|y| \to \infty$.

Proof

This theorem is proved in any standard textbook. We write

$$g(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx = -\int_{-\infty}^{\infty} f\left(x + \frac{1}{2y}\right) e^{-2\pi ixy}\,dx \tag{4.2}$$

by simple substitution like $x + \frac{1}{2y} = X$, and we have $\int_{-\infty}^{\infty} f(X)e^{-2\pi iXy}e^{\pi i}\,dX = -\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$. Hence

$$\begin{aligned} |g(y)| &= \left| \frac{1}{2}\int_{-\infty}^{\infty} \left\{ f(x) - f\left(x + \frac{1}{2y}\right)\right\} e^{-2\pi ixy}\,dx \right| \\ &\leq \frac{1}{2}\int_{-\infty}^{\infty} \left| f(x) - f\left(x + \frac{1}{2y}\right)\right|\,dx, \end{aligned} \tag{4.3}$$

which tends to zero as $|y| \to \infty$ by a fundamental theorem of integration. To use Theorem 4.1 one must be able to recognize absolute integrability in commonly occurring functions. The most useful test is as follows.

Example 1

If $f(x)$ is continuous except at $x = x_1, x = x_2, \ldots, x = x_M$, and if

$$f(x) = O(|x - x_m|^{\beta_m}) \quad \text{as } x \to x_m, \ \text{ where } \beta_m > -1, \tag{4.4}$$

for $m = 1, 2, 3, \ldots, M$, and

$$f(x) = O(|x|^{\beta_0}) \quad \text{as } x \to \infty, \ \text{ where } \beta_0 < -1, \tag{4.5}$$

then $f(x)$ is absolutely integrable from $-\infty$ to ∞ and therefore its Fourier transform $g(y) \to 0$ as $|y| \to \infty$.

The proof is immediate from the integrability properties of the comparison functions.

Example 2

Show that the Fourier transform of the function $f(x) = |x^4 - 1|^{-\frac{1}{2}}$ which is $g(y)$ tends to 0 as $|y| \to \infty$.

Proof

This function satisfies the conditions of Example 1, with $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $\beta_1 = -\frac{1}{2}$, $\beta_2 = 0$, $\beta_3 = -\frac{1}{2}$ and $\beta_0 = -2$. Hence its Fourier transform $g(y) \to 0$ as $|y| \to \infty$.

4.3 Generalization of the Riemann–Lebesgue lemma

For generalized functions, we need results similar to Theorem 4.1. In the following we cite some rather trivial definitions.

Definition 1

If $g(y)$ is a generalized function, then any statement like

$$g(y) \to 0,\ g(y) = O\{h(y)\} \text{ or } g(y) = o\{h(y)\}, \tag{4.6}$$

as $|y| \to c$ (or as $|y| \to \infty$), means that $g(y)$ is equal in some interval including $y = c$ (or in some interval $|y| > R$) to an ordinary function $g_1(y)$ satisfying the same condition.

Example 3

A simple and trivial example is $\delta(y) + \frac{1}{y} \to 0$ as $|y| \to \infty$.

Definition 2

If $f(x)$ is a generalized function which equals an ordinary function $f_1(x)$ in some interval $a < x < b$, and $f_1(x)$ is absolutely integrable in the interval (a, b), then we say that $f(x)$ is absolutely integrable in (a, b).

Example 4

A trivial example is that $\delta(x)$ is absolutely integrable in $(0, \infty)$ and $(-\infty, 0)$, but not in $(-\infty, \infty)$.

Theorem 4.2

If a generalized function $f(x)$ is absolutely integrable in $(-\infty, \infty)$ and $g(y)$ is its Fourier transform, then $g(y) \to 0$ as $|y| \to \infty$.

Proof

This, by Definitions 1 and 2, is just a rewritten version of the Riemann–Lebesgue lemma.

It would be worth noting that it will be wrong to conclude that absolutely any generalized function which is absolutely integrable in every finite interval has its Fourier transform tending to 0 as $|y| \to \infty$.

Example 5

Show that the Fourier transform of e^{ix^2} is $e^{-i\pi^2 y^2}(1+i)\sqrt{(\frac{\pi}{2})}$, which does not go to zero as $|y| \to \infty$.

Proof

The Fourier transform of the ordinary function $e^{(i-\varepsilon)x^2}$, of which e^{ix^2} is easily seen to be the limit as $\varepsilon \to 0$, is

$$\begin{aligned}\mathcal{F}\{e^{(i-\varepsilon)x^2}\} &= \int_{-\infty}^{\infty} e^{(i-\varepsilon)x^2 - 2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} e^{(i-\varepsilon)[(x-\frac{\pi iy}{i-\varepsilon})^2 - (\frac{\pi iy}{i-\varepsilon})^2]}\,dx \\ &= e^{\pi^2y^2/(i-\varepsilon)}\sqrt{\left(\frac{\pi}{\varepsilon - i}\right)} \\ &= e^{-i\pi^2y^2}(1+i)\sqrt{\left(\frac{\pi}{2}\right)},\end{aligned} \tag{4.7}$$

which is the limit as $\varepsilon \to 0$ is the function stated.

However, the function of Example 5 is somewhat exceptional in that it oscillates with a frequency which itself increases to infinity as $|x| \to \infty$. Most functions occurring in applications do not do this. This point is emphasized by the following definition and theorem which represent an attempt to include most of them in a general statement without making the latter too complicated to prove.

Definition 3

The generalized function $f(x)$ is said to be "well behaved at infinity" if for some R the function $f(x) - F(x)$ is absolutely integrable in the interval $(-\infty, -R)$ and (R, ∞), where $F(x)$ is some linear combination of the functions

$$e^{ikx}|x|^\beta,\ e^{ikx}|x|^\beta\, sgn(x),\ e^{ikx}|x|^\beta \ln|x|,\ e^{ikx}|x|^\beta \ln|x|\, sgn(x), \tag{4.8}$$

for different values of β and k.

It is to be noted that obviously no values of $\beta < -1$ need to be present in $F(x)$.

Theorem 4.3

If the generalized function $f(x)$ is well behaved at infinity and absolutely integrable in every finite interval, and $g(y)$ is its Fourier transform, then $g(y) \to 0$ as $|y| \to \infty$.

For proof of this theorem, the reader is referred to the work of Lighthill (1964).

Example 6

If $g(y)$ is the Fourier transform of $f(x) = |x|^\nu J_\nu(|x|)$, where $J_\nu(x)$ is the Bessel function of the first kind and of order ν, then $g(y) \to 0$ as $|y| \to \infty$ if $\nu > -\frac{1}{2}$.

Proof

$f(x)$ is continuous except at $x=0$, where it is of $O(|x|^{2\nu}|)$, and so it satisfies eqn (4.4) if $\nu > -\frac{1}{2}(2\nu > -1)$. Hence it is absolutely integrable in every finite interval. It is not absolutely integrable up to infinity, but, by asymptotic expansion for J_ν

$$f(x) = F(x) + O(|x|^{\nu-1/2-N}) \quad \text{as } |x| \to \infty,$$

where

$$F(x) = \frac{|x|^{\nu-1/2}}{\sqrt{(2\pi)}} \sum_{0}^{N-1} \frac{(\nu+n-\frac{1}{2})}{n!(\nu-n-\frac{1}{2})!(2i|x|)^n} \times \{(-1)^n e^{i(|x|-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} + e^{-i(|x|-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\} \tag{4.9}$$

is a linear combination of functions of type (4.8). It follows that $f(x)$ is well behaved at infinity, since if $N > \nu + \frac{1}{2}$, then $f(x) - F(x)$ satisfies eqn (4.5) and so is absolutely integrable up to infinity. Hence by Theorem 4.3, $g(y) \to 0$ as $|y| \to \infty$.

We now check this conclusion by calculating the Fourier transform $g(y)$. Note first that, if $\nu > -1$, $\lim_{\varepsilon\to 0}\{e^{-\varepsilon|x|}f(x)\} = f(x)$, since for any good function $F(x)$ we have

$$\left|\int_{-\infty}^{\infty} (1-e^{-\varepsilon|x|})|x|^{\nu} J_\nu(|x|)F(x)\,dx\right| \leq \varepsilon \int_{-\infty}^{\infty} |x|^{\nu+1}|J_\nu(|x|)F(x)|\,dx = O(\varepsilon) \tag{4.10}$$

if $\nu > -1$. Here $J_\nu(|x|)$ is the Bessel function of first kind with order ν.

Now with reference to the book by Watson (1944, 13.2, eqn (5)), the Fourier transform of $e^{-\varepsilon|x|}f(x)$ is

$$\mathcal{F}\{e^{-\varepsilon|x|}f(x)\} = \frac{2^\nu(\nu-\frac{1}{2})!}{[1+(\varepsilon+2\pi iy)^2]^{\nu+1/2}\sqrt{\pi}} + \frac{2^\nu(\nu-\frac{1}{2})!}{[1+(\varepsilon-2\pi iy)^2]^{\nu+1/2}\sqrt{\pi}}. \tag{4.11}$$

Hence, as $\varepsilon \to 0$, we have the Fourier transform $g(y)$ in the following manner:

$$g(y) = \mathcal{F}\{f(x)\} = \frac{2^{\nu+1}(\nu-\frac{1}{2})!}{[1-4\pi^2y^2]^{\nu+1/2}\sqrt{\pi}} \quad \left(|y| < \frac{1}{2\pi}\right), \tag{4.12}$$

$$g(y) = \mathcal{F}\{f(x)\} = -\frac{2^{\nu+1}(\nu-\frac{1}{2})!\sin\nu\pi}{[4\pi^2y^2-1]^{\nu+1/2}\sqrt{\pi}} \quad \left(|y| > \frac{1}{2\pi}\right). \tag{4.13}$$

This checks that $g(y)$ does tend to 0 as $|y| \to \infty$ when $\nu > -\frac{1}{2}$; and the fact that $g(y)$ does not tend to 0 for $-1 < \nu \leq -\frac{1}{2}$ reconfirms the importance of the condition that $f(x)$ be absolutely integrable in every finite interval.

4.4 The asymptotic expression of the Fourier transform of a function with a finite number of singularities

We will devote a considerable amount of effort to demonstrate the asymptotic expansion of the Fourier transform of a function with a finite number of singularities. We start with a definition below.

Definition 4

A generalized function $f(x)$ is said to have a finite number of singularities $x = x_1, x_2, x_3, \ldots, x_M$ if, in each one of the intervals $-\infty < x < x_1, x_1 < x < x_2, \ldots,$ $x_{M-1} < x < x_M, x_M < x < \infty$, $f(x)$ is equal to an ordinary function differentiable any number of times at every point of the interval.

Example 7

The function $f(x) = \delta''(x) + |x^4 - 5x^2 + 4|^{-3/2}$ has the singularities $x = -2$, -1, 0, 1 and 2. The singularity $x = 0$ arises from the delta function.

It is worth noting here that most ordinary or generalized functions which occur in practical applications have only a finite number of singularities; for these, the method of the present section is very important and very effective. The principal exceptions are periodic functions, which are treated separately in the next chapter.

Theorem 4.4

If the generalized function $f(x)$ has a finite number of singularities $x = x_1$, $x_2, \ldots, x_M$, and if (and for each m from 1 to M) $f(x) - F_m(x)$ has absolutely integrable Nth derivative in an interval including x_m, where $F_m(x)$ is a linear combination of the type

$$|x - x_m|^\beta, |x - x_m|^\beta \, sgn(x - x_m), |x - x_m|^\beta \ln|x - x_m|,$$
$$|x - x_m|^\beta \ln|x - x_m| \, sgn(x - x_m) \tag{4.14}$$

and $\delta^{(p)}(x - x_m)$, for different values of β and p, and if $f^{(N)}(x)$ is well behaved at infinity, then $g(y)$, the Fourier transform of $f(x)$, satisfies

$$g(y) = \sum_{m=1}^{M} G_m(y) + o(|y|^{-N}) \quad \text{as } |y| \to \infty, \tag{4.15}$$

where $G_m(y)$, the Fourier transform of $F_m(x)$, can be obtained from Formulae 1–18.

Proof

Let us consider that $f_R(x) = f(x) - \sum_{m=1}^{M} F_m(x)$, and $g_R(y)$ is the Fourier transform of $f_R(x)$. It can be easily shown that $\mathcal{F}\{f_R^{(N)}(x)\} = (2\pi i y)^N g_R(y)$. The calculations are as follows:

$$\mathcal{F}\{f_R(x)\} = \int_{-\infty}^{\infty} e^{-2\pi ixy} f_R(x)\,dx = g_R(y).$$

Now the inverse transform is

$$f_R(x) = \int_{-\infty}^{\infty} g_R(y) e^{2\pi ixy}\,dy.$$

Differentiating both sides N times with respect to x yields

$$f_R^{(N)}(x) = \int_{-\infty}^{\infty} [(2\pi iy)^N g_R(y)] e^{frm-e\pi ixy}\,dy = \mathcal{F}^{-1}\{(2\pi iy)^N g_R(y)\},$$

$$\mathcal{F}\{f_R^{(N)}(x)\} = (2\pi iy)^N g_R(y).$$

To prove eqn (4.15) we must show that $(2\pi iy)^N g_R(y)$ tends to 0 as $|y| \to \infty$. Now, $f_R^{(N)}(x)$ is absolutely integrable in the interval including x_m but no other singularities, because $f^{(N)}(x) - F_m^{(N)}(x)$ is, and so are $F_1^{(N)}(x), \ldots, F_{m-1}^{(N)}(x), F_{m+1}^{(N)}(x), \ldots, F_M^{(N)}(x)$. This being a correct conclusion for $m = 1, \ldots, M$, it follows that $f_R^{(N)}(x)$ is absolutely integrable in every finite interval; also, it is well behaved at infinity, since $f^{(N)}(x)$ is given to be, and each component in each of the $F_m^{(N)}(x)$ obviously is. Hence, by Theorem 4.3, the Fourier transform $(2\pi iy)^N g_R(y)$ of $f_R^{(N)}(x)$ tends to zero as $|y| \to \infty$, as stated in eqn (4.15).

Remark

The result of Theorem 4.4 is most often useful when $f(x)$ is an ordinary function. Note, however, that even in this case the statement (4.15) of the result, let alone its proof, would be meaningless outside generalized-function theory, since in ordinary Fourier transform theory the transforms $G_m(y)$ would exist only if all the $F_m(x)$ were composed solely of functions of the type (4.14) with $-1 < \beta < 0$.

The method of Theorem 4.4 will now be illustrated by a number of examples. After studying these, we shall practise the method on several of the exercises at the end of the chapter. It is instructive to begin with an example to which we already know the answer.

Example 8

Find an asymptotic expression for the Fourier transform of

$$f(x) = |x|^{\nu} J_{\nu}(|x|).$$

Solution

We recognize that the behaviour of $f(x)$ near its only singularity $x = 0$ is given by the series

$$f(x) = |x|^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}|x|)^{\nu+2n}}{n!(\nu+n)!}, \tag{4.16}$$

where

$$J_\nu(|x|) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}|x|)^{\nu+2n}}{n!(\nu+n)!}.$$

Hence if we consider that corresponding to $n = 0$

$$F_1(x) = \frac{|x|^{2\nu}}{2^\nu \nu!} \tag{4.17}$$

is the leading term of this series, then $f(x) - F_1(x)$ is of $O(|x|^{2\nu+2})$ as $x \to 0$, and its Nth derivative is absolutely integrable in any interval including the origin $x = 0$ if N is the least integer $\geq 2\nu + 2$. Also, $f(x)$ and its derivatives are well behaved at infinity, and so, by Theorem 4.4 and Formulae 1–18,

$$\begin{aligned} g(y) = G_1(y) + o(|y|^{-N}) &= \frac{[2\cos\frac{1}{2}\pi(2\nu+1)](2\nu)!}{\nu!2^\nu(2\pi|y|)^{2\nu+1}} + o(|y|^{-N}) \\ &= -\frac{2^{\nu+1}(\nu-\frac{1}{2})!\sin\nu\pi}{(2\pi|y|)^{2\nu+1}\sqrt{\pi}} + o(|y|^{-N}), \end{aligned} \tag{4.18}$$

where $g(y)$ and $G_1(y)$ are the Fourier transforms of $f(x)$ and $F_1(x)$, respectively, and the "duplication formula"

$$(2\nu)! = \nu!\left(\nu - \frac{1}{2}\right)!2^{2\nu}\pi^{-1/2} \tag{4.19}$$

has been used to throw the result into a form which can be immediately checked from the exact form (4.12) or (4.13) of $g(y)$.

Remark

It is worth noting in this situation that, by including more terms of the series (4.16) in $F_1(x)$, we could make higher derivatives (the $(N+1)$th, the $(N+2)$th and so on) of $f(x) - F_1(x)$ absolutely integrable in an interval including $x = 0$, and so reduce the error in the equation $g(y) + G_1(y)$ for large $|y|$ successively to $o(|y|^{N-2})$, $o(|y|^{N-4})$ and so on, depending on how many terms were included. In this way we would build up an "asymptotic expansion" of $g(y)$, which in the particular case here discussed would be simply the binomial expansion of

$$g(y) = \mathcal{F}\{f(x)\} = -\frac{2^{\nu+1}(\nu - \frac{1}{2})! \sin \nu\pi}{[4\pi^2 y^2 - 1]^{\nu+1/2}\sqrt{\pi}} \quad \left(|y| > \frac{1}{2\pi}\right)$$

in descending powers of y.

Example 9

Find an asymptotic expansion for the Fourier transform of $f(x) = |x||x+1|^{1/2}|x-1|^{-1/2}$ with an error of $o(|y|^{-2})$.

Solution

The singularities of $f(x)$ are at $x = -1$, 0 and $+1$. We need to express $f(x)$ near each singularity as a sum of terms like eqn (4.14) with an error whose second derivative is absolutely integrable in an interval including the singularity. Thus we illustrate the expansion of $f(x)$ by binomial series about each singular point.

Expanding $f(x)$ about $x = -1$ we obtain

$$\begin{aligned}
f(x) &= |x+1|^{1/2}[|(x+1)-1|] \times [|(x+1)-2|]^{-1/2} \\
&= \frac{1}{\sqrt{2}}|x+1|^{1/2}[|1-(x+1)|] \times \left[\left|1 - \frac{(x+1)}{2}\right|\right]^{-1/2} \\
&= \frac{1}{\sqrt{2}}|x+1|^{1/2}[|1-(x+1)|] \times \left[\left|1 + \frac{x+1}{4}\right| + \cdots\right] \\
&= \frac{1}{\sqrt{2}}|x+1|^{1/2}\left[1 - \frac{3}{4}|x+1| + \cdots\right] \\
&= \frac{1}{\sqrt{2}}|x+1|^{1/2} + O(|x+1|^{3/2}).
\end{aligned}$$

Expanding $f(x)$ about $x = 0$ yields

$$\begin{aligned}
f(x) &= |x|\left[1 + \left|\frac{x}{2}\right| + \cdots\right] \times \left[1 + \left|\frac{x}{2}\right| + \cdots\right] \\
&= |x| + O(|x|^2).
\end{aligned}$$

Expanding $f(x)$ about $x = +1$ we obtain

$$\begin{aligned}
f(x) &= |x-1|^{-1/2}[|(x+1)-1|] \times [|(x-1)+2|]^{1/2} \\
&= \sqrt{2}|x-1|^{-1/2}\left[1 + \frac{5}{4}|(x+1)| - \frac{1}{32}|x-1|^2 + \cdots\right] \\
&= \sqrt{2}|x-1|^{-1/2} + \frac{5}{2\sqrt{2}}|x-1|^{1/2}\, sgn(x-1) + O(|x-1|^{3/2}).
\end{aligned}$$

Let us define the functions

$$F_1(x) = \frac{1}{\sqrt{2}}|x+1|^{1/2},$$
$$F_2(x) = |x|,$$
$$F_3(x) = \sqrt{2}|x-1|^{-1/2} + \frac{5}{2\sqrt{2}}|x-1|^{1/2}\, sgn(x-1).$$

We see that the conditions of Theorem 4.4 with $N = 2$ are satisfied, and therefore we can write the Fourier transforms as follows:

$$\begin{aligned}
g(y) &= G_1(y) + G_2(y) + G_3(y) + o(|y|^{-2}) \\
&= e^{2\pi iy}\frac{1}{\sqrt{2}}\frac{-\sqrt{(\frac{1}{2}\pi)}}{(2\pi|y|)^{3/2}} + \frac{2}{(2\pi iy)^2} \\
&\quad + e^{-2\pi iy}\left\{\sqrt{2}\frac{\sqrt{2\pi}}{(2\pi|y|)^{1/2}} + \frac{5}{2\sqrt{2}}\frac{(-i\, sgn(y))\sqrt{(\frac{1}{2}\pi)}}{(2\pi|y|)^{3/2}}\right\} + o(|y|^{-2}) \\
&= (\sqrt{2})e^{-2\pi iy}|y|^{-1/2} - \left\{\frac{1}{4\pi\sqrt{2}}e^{2\pi iy} + \frac{5i\, sgn(y)}{8\pi\sqrt{2}}e^{-2\pi iy}\right\}|y|^{-3/2} \\
&\quad - \frac{1}{2\pi^2}y^{-2} + o(|y|^{-2}).
\end{aligned}$$

As in Example 8, if larger terms in the expansions of $f(x)$ near each of its singularities were retained in $F_m(x)$, then the error term in the expansion of $g(y)$ could be reduced in magnitude for large $|y|$. We could build up in this way $g(y)$ for large $|y|$ as the sum of three asymptotic series, one in simple power of y, one in powers with the factor $e^{2\pi iy}$ outside and one in powers with factor $e^{-2\pi iy}$ outside. The largest term in $g(y)$, that is, the one asymptotically largest for large $|y|$, is that in $|y|^{-1/2}$, which arises from the term in $f(x)$ which is of order $|x-1|^{-1/2}$ as $x \to 1$.

Example 10

Find an asymptotic expression for

$$g(y) = \int_0^1 \frac{e^{-2\pi ixy}\cosh x}{(1-x^4)^{1/2}}\, dx, \tag{4.20}$$

with an error $o(|y|^{-2})$.

Solution

The given integral can be written as

$$\begin{aligned} g(y) &= \int_0^1 \frac{e^{-2\pi ixy}\cosh x}{(1-x^4)^{1/2}}\,dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-2\pi ixy}\cosh x}{(1-x^4)^{1/2}} H(x)H(1-x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= \mathcal{F}\{f(x)\}, \end{aligned}$$

where $f(x) = (1-x^4)^{-1/2}(\cosh x)H(x)H(1-x)$. Now the function $f(x)$ has the singularities at $x = 0$ and 1.

Near $x = 0$, $f(x) - F_1(x)$ has absolutely integrable second derivative, being in fact $O(|x|^2)$, if

$$F_1(x) = H(x).$$

Near $x = 1$, $f(x)$ can be expanded by Taylor series and we obtain

$$\begin{aligned} f(x) &= \frac{\cosh 1 + (x-1)\sinh 1}{(1-x)^{1/2}[2 - \frac{3}{2}(1-x)]} H(1-x) + O(1-x)^{3/2} \\ &= F_2(x) + O(1-x)^{3/2}, \end{aligned}$$

where

$$\begin{aligned} F_2(x) &= \left(\frac{1}{2}\cosh 1\right)(1-x)^{-1/2}H(1-x) \\ &\quad + \left(\frac{3}{8}\cosh 1 - \frac{1}{2}\sinh 1\right)(1-x)^{1/2}H(1-x). \end{aligned}$$

Applying Theorem 4.4 with $N = 2$, we deduce using Formulae 1–18 that

$$\begin{aligned} g(y) = \frac{1}{2\pi iy} + e^{-2\pi iy}\Bigg[&\left(\frac{1}{2}\cosh 1\right)\frac{e^{\frac{1}{4}\pi i\, sgn(y)}\sqrt{\pi}}{(2\pi|y|)^{1/2}} \\ &+ \left(\frac{3}{8}\cosh 1 - \frac{1}{2}\sinh 1\right)\frac{e^{\frac{3}{4}\pi i\, sgn(y)}(\frac{1}{2}\sqrt{\pi})}{(2\pi|y|)^{3/2}}\Bigg] + o(|y|^{-2}) \end{aligned}$$

as $|y| \to \infty$. From the detailed expressions for the errors in $F_1(x)$ at $x = 0$ and in $F_2(x)$ at $x = 1$, we can say that the precise order of magnitude of the error in the above result is $O(|y|^{-3/2})$.

Remark

$f(x) = (1-x^4)^{-1/2}(\cosh x)H(x)H(1-x)$. The function has two singularities $x=0$ and 1. By Taylor's expansion about each of the singularities we have in principle

$$
\begin{aligned}
f(x) &= H(x)f_1(x) = H(x)\left[f_1(0) + xf_1'(0) + \frac{x^2}{2!}f_1''(0) + \cdots\right], \quad \text{about } x=0 \\
&= H(x) + O(|x|^2), \\
f(x) &= (1-x)^{-1/2}H(1-x)f_2(x) \\
&= (1-x)^{-1/2}H(1-x)\left[f_2(1) + (x-1)f_2'(1) + \frac{1}{2!}(x-1)^2 f_2''(1) + \cdots\right] \\
&\quad \text{about } x = 1 \\
&= \left(\frac{1}{2}\cosh 1\right)(1-x)^{-1/2}H(1-x) \\
&\quad + \left(\frac{3}{8}\cosh 1 - \frac{1}{2}\sinh 1\right)(1-x)^{1/2}H(1-x) + O(|x-1|^{3/2}).
\end{aligned}
$$

Example 11

If $F(x)$ and all its derivatives exist as ordinary functions for $x \geq 0$, and are well behaved at infinity, derive the asymptotic expansion

$$
\int_0^\infty F(x)\sin 2\pi xy\,dx \sim \frac{F(0)}{2\pi y} - \frac{F''(0)}{(2\pi y)^3} + \frac{F^{iv}(0)}{(2\pi y)^5} - \cdots . \tag{4.21}
$$

Solution

(a) Direct integration by parts

$$
\begin{aligned}
\int_0^\infty F(x)\sin 2\pi xy\,dx &= \frac{F(0)}{2\pi y} + \frac{1}{2\pi y}\int_0^\infty F'(x)\cos 2\pi xy\,dx \\
&= \frac{F(0)}{2\pi y} - \frac{1}{(2\pi y)^2}\int_0^\infty F''(x)\sin 2\pi xy\,dx \\
&= \frac{F(0)}{2\pi y} - \frac{F''(0)}{(2\pi y)^3} - \frac{1}{(2\pi y)^3}\int_0^\infty F'''(x)\cos 2\pi xy\,dx \\
&= \frac{F(0)}{2\pi y} - \frac{F''(0)}{(2\pi y)^3} + \frac{F^{iv}(0)}{(2\pi y)^5} - \cdots .
\end{aligned}
$$

This result is the same as above. By using the asymptotic expansion method we see that eqn (4.21) is nothing but the "half-range" Fourier sine integral which signifies $\frac{1}{2}ig(y)$, where $g(y)$ is the Fourier transform of

$$f(x) = F(|x|)\, sgn(x). \tag{4.22}$$

Now $f(x)$ has only one singularity, $x = 0$, near which $f(x) - F_1(x)$ has absolutely integrable $(2p)$th derivative if

$$F_1(x) = \left\{ F(0) + \frac{F''(0)}{2!}x^2 + \cdots + \frac{F^{(2p-2)}}{(2p-2)!}x^{2p-2} \right\} sgn(x). \tag{4.23}$$

Hence the conditions of Theorem 4.4 with $N = 2p$ are satisfied, whence, using Formulae 1–18,

$$g(y) = \sum_{n=0}^{p-1} \frac{F^{(2n)}(0)}{(2n)!} 2 \frac{(2n)!}{(2\pi iy)^{2n+1}} + o(|y|^{-2p})$$

as $|y| \to \infty$. The fact that this is true for all p is what is meant by the "asymptotic expansion" formula (4.21) for $\frac{1}{2}ig(y)$.

Similarly, under the same conditions, we have

$$\int_0^\infty F(x) \cos 2\pi xy\, dx \sim -\frac{F'(0)}{(2\pi y)^2} + \frac{F'''(0)}{(2\pi y)^4} - \frac{F^v(0)}{(2\pi y)^6} + \cdots .$$

(b) Direct integration by parts

$$\begin{aligned}
\int_0^\infty F(x) \cos 2\pi xy\, dx &= -\frac{1}{2\pi y} \int_0^\infty F'(x) \sin 2\pi xy\, dx \\
&= -\frac{F'(0)}{(2\pi y)^2} - \frac{1}{(2\pi y)^2} \int_0^\infty F''(x) \cos 2\pi xy\, dx \\
&= -\frac{F'(0)}{(2\pi y)^2} + \frac{F'''(0)}{(2\pi y)^4} + \frac{1}{(2\pi y)^4} \int_0^\infty F^{iv}(x) \cos 2\pi xy\, dx \\
&= -\frac{F'(0)}{(2\pi y)^2} + \frac{F'''(0)}{(2\pi y)^4} - \frac{F^v(0)}{(2\pi y)^6} + \cdots .
\end{aligned}$$

This result is the same as above.

Remark

We need to show that $\int_0^\infty F(x)\sin 2\pi xy\,dx = \frac{i}{2}\int_{-\infty}^\infty f(x)e^{-2\pi ixy}\,dx = \frac{i}{2}g(y)$, where $f(x) = F(|x|)\,sgn(x)$.

$$\begin{aligned}\int_0^\infty F(x)\sin 2\pi xy\,dx &= \frac{1}{2i}\int_0^\infty F(x)[e^{2\pi ixy} - e^{-2\pi ixy}]\,dx\\ &= \frac{-i}{2}\left[\int_0^\infty F(x)e^{2\pi ixy}\,dx - \int_0^\infty F(x)e^{-2\pi ixy}\,dx\right]\\ &= \frac{i}{2}\left[-\int_{-\infty}^0 F(-x)e^{-2\pi ixy}\,dx + \int_0^\infty F(x)e^{-2\pi ixy}\,dx\right]\\ &= \frac{i}{2}\int_{-\infty}^\infty F(|x|)\,sgn(x)e^{-2\pi ixy}\,dx\\ &= \frac{i}{2}\int_{-\infty}^\infty f(x)e^{-2\pi ixy}\,dx\\ &= \frac{i}{2}g(y).\end{aligned}$$

Here $f(x) = F(|x|)\,sgn(x)$.

Similarly we have for the integral $\int_0^\infty F(x)\cos 2\pi xy\,dx$

$$\begin{aligned}\int_0^\infty F(x)\cos 2\pi xy\,dx &= \frac{1}{2}\int_0^\infty F(x)[e^{2\pi ixy} + e^{-2\pi ixy}]\,dx\\ &= \frac{1}{2}\left[\int_0^\infty F(x)e^{2\pi ixy}\,dx + \int_0^\infty F(x)e^{-2\pi ixy}\,dx\right]\\ &= \frac{1}{2}\left[\int_{-\infty}^0 F(-x)e^{-2\pi ixy}\,dx + \int_0^\infty F(x)e^{-2\pi ixy}\,dx\right]\\ &= \frac{1}{2}\int_{-\infty}^\infty F(|x|)e^{-2\pi ixy}\,dx\\ &= \frac{1}{2}\int_{-\infty}^\infty f(x)e^{-2\pi ixy}\,dx\\ &= \frac{1}{2}g(y).\end{aligned}$$

Here $f(x) = F(|x|)$.

Example 12

Show that $g(y) = \int_0^1 K_0(x)\cos 2\pi xy\,dx = \mathcal{F}\{f(x)\}$, where $f(x) = \frac{1}{2}K_0(|x|)H(x+1)$ $H(1-x)$. Here $K_0(x)$ is the modified Bessel function of the second kind and of order zero.

Proof

The proof is given below.

$$\begin{aligned}
g(y) &= \int_0^1 K_0(x)\cos 2\pi xy\,dx \\
&= \frac{1}{2}\left\{\int_0^1 K_0(x)e^{2\pi ixy}\,dx + \int_0^1 K_0(x)e^{-2\pi ixy}\,dx\right\} \\
&= \frac{1}{2}\left\{\int_0^{-1} K_0(-x)e^{-2\pi ixy}(-dx) + \int_0^1 K_0(x)e^{-2\pi ixy}\,dx\right\} \\
&= \frac{1}{2}\left\{\int_{-1}^{0} K_0(-x)e^{-2\pi ixy}\,dx + \int_0^1 K_0(x)e^{-2\pi ixy}\,dx\right\} \\
&= \frac{1}{2}\left\{\int_{-1}^{1} K_0(|x|)e^{-2\pi ixy}\,dx\right\} \\
&= \frac{1}{2}\left\{\int_{-\infty}^{\infty} K_0(|x|)e^{-2\pi ixy}H(x+1)H(1-x)\,dx\right\} \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\
&= \mathcal{F}\{f(x)\},
\end{aligned}$$

where $f(x) = \frac{1}{2}K_0(|x|)H(x+1)H(1-x)$. Hence this is the required proof.

Example 13

Find an asymptotic expression for

$$g(y) = \int_0^1 K_0(x)\cos 2\pi xy\,dx \tag{4.24}$$

with an error $o(|y|^{-1})$, where $K_0(x)$ is the modified Bessel function of the second kind of order zero.

Solution

This $g(y)$ is the Fourier transform of

$$f(x) = \frac{1}{2}K_0(|x|)H(x+1)H(1-x),$$

which has singularities $x = -1$, 0 and $+1$, whence

$$f(x) = \frac{1}{2}K_0(1)H(x+1) + O(|x+1|),$$
$$f(x) = \frac{1}{2}\left[-\ln\left(\frac{1}{2}|x|\right) - \gamma\right] + O(|x|^2 \ln |x|),$$
$$f(x) = \frac{1}{2}K_0(1)H(1-x) + O(|x-1|),$$

respectively. Hence, by Theorem 4.4 with $N = 1$ and Formulae 1–18,

$$\begin{aligned} g(y) &= \frac{1}{2}K_0(1)\frac{e^{2\pi iy}}{2\pi iy} - \frac{1}{2}\left(-\frac{sgn(y)}{2y}\right) + \frac{1}{2}K_0(1)\frac{e^{-2\pi iy}}{-2\pi iy} + o(|y|^{-1}) \\ &= K_0(1)\frac{\sin 2\pi y}{2\pi y} + \frac{1}{4|y|} + O(|y|^{-2}), \end{aligned}$$

where the precise form of the error term is $O(|y|^{-2})$ because the worst error term is of the $O(|x - x_m|)$ form.

Example 14

Find an asymptotic expression for the Fourier transform of $e^{-|x|}$ with an error $o(|y|^{-3})$, and check it against the exact expression obtained by direct integration.

Solution

Direct integration yields the following transform:

$$\begin{aligned} \mathcal{F}\{e^{-|x|}\} &= \int_{-\infty}^{\infty} e^{-|x|}e^{-2\pi ixy}\,dx \\ &= \int_{-\infty}^{0} e^{(1-2\pi iy)x}\,dx + \int_{0}^{\infty} e^{-(1+2\pi iy)x}\,dx \\ &= \frac{1}{1-2\pi iy} + \frac{1}{1+2\pi iy} \\ &= \frac{2}{1+4\pi^2y^2} = g(y). \end{aligned}$$

We can obtain an asymptotic expression for the Fourier transform of $e^{-|x|}$. This function is an even function and so we have the half-range Fourier cosine transform as follows:

$$\begin{aligned}\mathcal{F}\{e^{-|x|}\} &= 2\int_0^\infty e^{-x}\cos 2\pi xy\,dx\\ &= 2\left[\frac{1}{(2\pi y)^2} - \frac{1}{(2\pi y)^4} + \frac{1}{(2\pi y)6} - \cdots\right]\\ &= \frac{2}{(2\pi y)^2}\left[1 + \frac{1}{(2\pi y)^2}\right]^{-1}\\ &= \frac{2}{1+4\pi^2y^2},\end{aligned}$$

which coincides with the above result by direct integration.

Note that

$$\int_0^\infty F(x)\cos 2\pi xy\,dx \sim -\frac{F'(0)}{(2\pi y)^2} + \frac{F'''(0)}{(2\pi y)^4} - \frac{F^v(0)}{(2\pi y)^6} + \cdots .$$

Example 15

Derive the asymptotic expansion of

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi ixy}}{1+|x|^3}\,dx. \tag{4.25}$$

Solution

The function $\frac{1}{1+|x|^3}$ is an even function and hence we have the half-range Fourier cosine integral as shown below.

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{1+|x|^3}\right\} &= \int_{-\infty}^{\infty}\frac{e^{-2\pi ixy}}{1+|x|^3}\,dx\\ &= 2\int_0^\infty \frac{\cos 2\pi xy}{1+x^3}\,dx\\ &\sim -2\frac{3!}{(2\pi y)^4} + 2\frac{9!}{(2\pi y)^{10}} - \cdots .\end{aligned}$$

Example 16

Find an asymptotic expression for the Fourier transform of $|x^4 - 5x^2 + 4|^{-1/2}\,sgn(x)$ with an error $o(|y|^{-1})$, and state the precise order of magnitude of the error.

Solution

$$\begin{aligned}\mathcal{F}\{|x^4-5x^2+4|^{-1/2}\,sgn(x)\} &= \int_{-\infty}^{\infty} |x^4-5x^2+4|^{-1/2}\,sgn(x)e^{-2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx,\end{aligned}$$

where

$$f(x) = |x^4-5x^2+4|^{-1/2}\,sgn(x) = |(x^2-4)(x^2-1)|^{-1/2}\,sgn(x).$$

The function $f(x)$ has singularities at $x=-2,-1,0,1$ and 2. Corresponding to each singularity we will have asymptotic expansions and subsequently the Fourier transforms. By Taylor's expansion, we will find the asymptotic form of each expansion of $f(x)$.

(a) Taylor's expansion of $f(x)$ about the singular point $x=-2$ is given below.

$$\begin{aligned}f(x) &= |x+2|^{-1/2}\{|(x-2)(x^2-1)|^{-1/2}\,sgn(x)\} \\ &= |x+2|^{-1/2}f_1(x),\end{aligned}$$

where

$$\begin{aligned}f_1(x) &= \{|(x-2)(x^2-1)|^{-1/2}sgn(x)\} \\ &= f_1(-2)+|(x+2)|\,f_1'(-2)+O(|x+2|^2) \\ &= -\frac{1}{2\sqrt{3}}+\frac{19}{48\sqrt{3}}|x+2|+O(|x+2|^2).\end{aligned}$$

Therefore we have

$$f(x) = -\frac{1}{2\sqrt{3}}|x+2|^{-1/2}+\frac{19}{48\sqrt{3}}|x+2|^{1/2}+O(|x+2|^{3/2}) = F_1(x).$$

(b) Taylor's expansion of $f(x)$ about the singular point $x=-1$ is given below.

$$\begin{aligned}f(x) &= |x+1|^{-1/2}\{|(x^2-4)(x-1)|^{-1/2}\,sgn(x)\} \\ &= |x+1|^{-1/2}f_2(x),\end{aligned}$$

where

$$\begin{aligned}f_2(x) &= \{|(x^2-4)(x-1)|^{-1/2}\,sgn(x)\} \\ &= f_2(-1)+|(x+1)|\,f_2'(-1)+O(|x+1|^2) \\ &= -\frac{1}{2\sqrt{6}}+\frac{1}{12\sqrt{6}}|x+1|+O(|x+1|^2).\end{aligned}$$

Therefore we have

$$f(x) = -\frac{1}{\sqrt{6}}|x+1|^{-1/2} + \frac{1}{12\sqrt{6}}|x+1|^{1/2} + O(|x+1|^{3/2}) = F_2(x).$$

(c) Taylor's expansion of $f(x)$ about the singular point $x=0$ is given below.

$$\begin{aligned} f(x) &= sgn(x)\{|(x^4-5x^2+4)|^{-1/2}\} \\ &= sgn(x)\, f_3(x), \end{aligned}$$

where

$$\begin{aligned} f_3(x) &= \{|(x^4-5x^2+4)|^{-1/2}\} \\ &= f_3(0) + |x|\, f_3'(0) + O(|x|^2) \\ &= -\frac{1}{2} + O(|x|^2). \end{aligned}$$

Therefore we have

$$f(x) = -\frac{1}{2} sgn(x) + O(|x|^2)\, sgn(x) = F_3(x).$$

(d) Taylor's expansion of $f(x)$ about the singular point $x=+1$ is given below.

$$\begin{aligned} f(x) &= |x-1|^{-1/2}\{|(x^2-4)(x+1)|^{-1/2}\, sgn(x)\} \\ &= |x-1|^{-1/2} f_4(x), \end{aligned}$$

where

$$\begin{aligned} f_4(x) &= \{|(x^2-4)(x+1)|^{-1/2}\, sgn(x)\} \\ &= f_4(+1) + |(x-1)|\, f_4'(+1) + O(|x-1|^2) \\ &= \frac{1}{\sqrt{6}} - \frac{1}{12\sqrt{6}}|x-1| + O(|x-1|^2). \end{aligned}$$

Therefore we have

$$f(x) = \frac{1}{\sqrt{6}}|x-1|^{-1/2} - \frac{1}{12\sqrt{6}}|x-1|^{1/2} + O(|x-1|^{3/2}) = F_4(x).$$

(e) Taylor's expansion of $f(x)$ about the singular point $x=+2$ is given below.

$$\begin{aligned} f(x) &= |x-2|^{-1/2}\{|(x+2)(x^2-1)|^{-1/2} sgn(x)\} \\ &= |x-2|^{-1/2} f_5(x), \end{aligned}$$

where

$$f_5(x) = \{|(x+2)(x^2-1)|^{-1/2}\, sgn(x)\} = f_5(+2) + |(x-2)|\, f_5'(+2) + O(|x-2|^2)$$
$$= \frac{1}{2\sqrt{3}} - \frac{19}{48\sqrt{3}}|x-2| + O(|x-2|^2).$$

Therefore we have

$$f(x) = \frac{1}{2\sqrt{3}}|x-2|^{-1/2} - \frac{19}{48\sqrt{3}}|x-2|^{1/2} + O(|x-2|^{3/2}) = F_5(x).$$

The following Table 4.1 displays some values of singular integral functions using numerical computations as illustrated here in this table.

Table 4.1: Numerical values of some singular integral functions.

x	$y = \frac{x^{-1/2}}{1+x}$	$y = \frac{x^{-3/2}}{1+x}$	$y = \frac{x^{-5/2}}{1+x}$
0.0	∞	∞	∞
0.1	2.8948	28.7479	287.4798
0.2	1.8633	9.3169	46.5847
0.3	1.4044	4.6813	15.6046
0.4	1.1294	2.8234	7.0586
0.5	0.9428	1.8856	3.7712
0.6	0.8069	1.0672	2.2413
0.7	0.7031	1.0044	1.4348
0.8	0.6211	0.7764	0.9705
0.9	0.5548	0.6164	0.6849
1.0	0.5000	0.5000	0.5000
1.5	0.3265	0.2177	0.1452
2.0	0.2357	0.1178	0.0589
2.5	0.1807	0.0722	0.0289
3.0	0.1443	0.0481	0.0160
3.5	0.1188	0.0339	0.0097
4.0	0.1000	0.0250	0.0062
4.5	0.0857	0.0190	0.0042
5.0	0.0745	0.0149	0.0029

We know that if $\mathcal{F}\{f(x)\} = g(y)$, then $\mathcal{F}\{f(x \pm x_0)\} = e^{\pm(2\pi i y x_0)}\mathcal{F}\{f(x)\} = e^{\pm(2\pi i y x_0)}g(y)$. Let us consider that $G_1(y), G_2(y), G_3(y), G_4(y)$ and $G_5(y)$ are the Fourier transforms of $F_1(x), F_2(x), F_3(x), F_4(x)$ and $F_5(x)$, respectively, where these F's are the asymptotic expansions of $f(x)$ at the singularities, then by Theorem 4.4 with $N = 2$, we can write (like residue calculus)

$$g(y) = G_1(y) + G_2(y) + G_3(y) + G_4(y) + G_5(y) + o(|y|^{-2}),$$

where

$$\begin{aligned}
G_1(y) &= -\frac{1}{2\sqrt{3}}e^{4\pi i y}\mathcal{F}\{|x|^{-1/2}\} + \frac{19}{48\sqrt{3}}e^{4\pi i y}\mathcal{F}\{|x|^{1/2}\},\\
G_2(y) &= -\frac{1}{\sqrt{6}}e^{2\pi i y}\mathcal{F}\{|x|^{-1/2}\} + \frac{1}{12\sqrt{6}}e^{4\pi i y}\mathcal{F}\{|x|^{1/2}\},\\
G_3(y) &= -\frac{1}{2}\mathcal{F}\{sgn(x)\},\\
G_4(y) &= \frac{1}{\sqrt{6}}e^{-2\pi i y}\mathcal{F}\{|x|^{-1/2}\} - \frac{1}{12\sqrt{6}}e^{-2\pi i y}\mathcal{F}\{|x|^{1/2}\},\\
G_5(y) &= \frac{1}{2\sqrt{3}}e^{-4\pi i y}\mathcal{F}\{|x|^{-1/2}\} - \frac{19}{48\sqrt{3}}e^{-4\pi i y}\mathcal{F}\{|x|^{1/2}\}.
\end{aligned}$$

We know from Formula 1 of Table 4.2 that

$$\begin{aligned}
\mathcal{F}\{|x|^{-1/2}\} &= -2\sqrt{\pi}(2\pi|y|)^{-1/2},\\
\mathcal{F}\{|x|^{1/2}\} &= -\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}.
\end{aligned}$$

Thus using these results and after a little reduction, we obtain

$$\begin{aligned}
G_1(y) + G_5(y) &= -\frac{i}{\sqrt{3}}\sin(4\pi y)[-2\sqrt{\pi}(2\pi|y|)^{-1/2}]\\
&\quad + \frac{19i}{24\sqrt{3}}\sin(4\pi y)\left[-\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}\right],\\
G_2(y) + G_4(y) &= -\frac{2i}{\sqrt{6}}\sin(2\pi y)[-2\sqrt{\pi}(2\pi|y|)^{-1/2}]\\
&\quad + \frac{i}{6\sqrt{6}}\sin(2\pi y)\left[-\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}\right],\\
G_3(y) &= -\frac{1}{2}\mathcal{F}\{sgn(x)\} = \frac{i}{2\pi}|y|^{-1}.
\end{aligned}$$

Table 4.2: Fourier transforms of some important generalized functions. In this table, α stands for any real number not an integer, n for any integer ≥ 0, m for any integer >0 and C for an arbitrary constant.

Formulae	$f(x)$	$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$
1	$\lvert x\rvert^{\alpha}$	$-2\sin(\frac{\pi}{2}\alpha)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$
2	$\lvert x\rvert^{\alpha}\, sgn(x)$	$-2i\cos(\frac{\pi}{2}\alpha)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\, sgn(y)$
3	$\lvert x\rvert^{\alpha}H(x)$	$\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}(e^{-\frac{1}{2}i(\alpha+1)\,sgn(y)})$
4	x^n	$(-2\pi i)^{-n}\delta^{(y)}$
5	$x^n\, sgn(x)$	$2n!(2\pi iy)^{-n-1}$
6	$x^nH(x)$	$(-2\pi i)^{-n}\left\{\frac{1}{2}\delta^{(y)} + \frac{(-1)^n n!}{2\pi iy^{n+1}}\right\}$
7	x^{-m}	$-\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}sgn(y)$
8	$x^{-m}\, sgn(x)$	$-2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln\lvert y\rvert + C)$
9	$x^{-m}H(x)$	$-\frac{(-2\pi iy)^{m-1}}{(m-1)!}[\frac{1}{2}\pi i\, sgn(y) + \ln\lvert y\rvert + C]$
10	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert$	$(2\cos\frac{1}{2}\pi(\alpha+1)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$ $\times(-2\ln(2\pi\lvert y\rvert) + \psi(\alpha) - \frac{1}{2}\tan\frac{1}{2}(\alpha+1))$
11	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert\, sgn(x)$	$(-2i\sin\frac{1}{2}\pi(\alpha+1))\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\, sgn(y)$ $\times(-2\ln(2\pi\lvert y\rvert) + \psi(\alpha) + \frac{1}{2}\cot\frac{1}{2}(\alpha+1))$
12	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert H(x)$	$(e^{-\frac{1}{2}\pi i(\alpha+1)\,sgn(y)})\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$ $\times\{-\ln(2\pi\lvert y\rvert) + \psi(\alpha) - \frac{1}{2}\pi i\, sgn(y)\}$
13	$x^n\ln\lvert x\rvert$	$-\pi i\frac{n!}{(2\pi iy)^{n+1}}sgn(y)$
14	$x^n\ln\lvert x\rvert\, sgn(x)$	$-2\frac{n!}{(2\pi iy)^{n+1}}[\ln(2\pi\lvert y\rvert) - \psi(n)]$
15	$x^n\ln\lvert x\rvert H(x)$	$-\frac{n!}{(2\pi iy)^{n+1}} \times [\frac{1}{2}\pi i\, sgn(y) + \ln(2\pi\lvert y\rvert) - \psi(n)]$
16	$x^{-m}\ln\lvert x\rvert$	$\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\, sgn(y) \times [\ln(2\pi\lvert y\rvert) - \psi(m-1)]$
17	$x^{-m}\ln\lvert x\rvert\, sgn(x)$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!} \times [(\ln(2\pi\lvert y\rvert) - \psi(m-1))^2 + C]$
18	$x^{-m}\ln\lvert x\rvert H(x)$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!}[\frac{1}{2}(\frac{1}{2}\pi i\, sgn(y) + \ln(2\pi\lvert y\rvert) - \psi(m-1))^2 + C]$

Hence the final result can be put together as follows:

$$g(y) = [2\sqrt{\pi}(2\pi|y|)^{-1/2}] \times \left[\frac{i}{\sqrt{3}}\sin(4\pi y) + \frac{2i}{\sqrt{6}}\sin(2\pi y)\right]$$
$$-\left[\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}\right] \times \left[\frac{19i}{24\sqrt{3}}\sin(4\pi y) + \frac{i}{6\sqrt{6}}\sin(2\pi y)\right]$$
$$+\frac{i}{2\pi}|y|^{-1} + o(|y|^{-2}).$$

It is worth mentioning that a considerable amount of algebraic calculations are necessary in obtaining this result. I hope the reader will get some benefit through this example in connection with the asymptotic expansion of a singular function like $f(x)$.

Example 17

Find an asymptotic expression for

$$\int_0^\infty (1-x)^{-1}\cos 2\pi xy\, dx. \tag{4.26}$$

Solution

We can use the formula for the asymptotic result as follows. Let $F(x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots$

$$\int_0^\infty (1-x)^{-1}\cos 2\pi xy\, dx \sim -\frac{1}{(2\pi y)^2} + \frac{3!}{(2\pi y)^4} - \frac{5!}{(2\pi y)^6} + \cdots.$$

Thus this is the asymptotic result.

Example 18

Find an asymptotic expression for

$$\int_0^1 \frac{\ln x}{(1-x)^{3/2}}\sin 2\pi xy\, dx \tag{4.27}$$

with an error $o(|y|^{-2})$, and state the precise order of magnitude of the error.

Solution

The given integral can be thrown into a Fourier integral form as follows:

$$\begin{aligned}
\int_0^1 \frac{\ln x}{(1-x)^{3/2}} \sin 2\pi xy\, dx &= \frac{1}{2i}\int_0^1 \frac{\ln x}{(1-x)^{3/2}}[e^{2\pi ixy} - e^{-2\pi ixy}]\, dx \\
&= \frac{1}{2i}\left[\int_{-1}^0 \frac{\ln(-x)}{(1-(-x))^{3/2}} e^{-2\pi ixy}\, dx \right.\\
&\qquad \left. - \int_0^1 \frac{\ln x}{(1-x)^{3/2}} e^{-2\pi ixy}\, dx\right] \\
&= \int_{-\infty}^{\infty} \frac{i}{2}\frac{\ln|x|}{|1-x|^{3/2}}\, sgn(x) e^{-2\pi ixy}\, dx \\
&= \frac{i}{2}\int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx \\
&= \frac{i}{2} g(y),
\end{aligned}$$

where

$$f(x) = \frac{\ln|x|}{|1-x|^{3/2}}\, sgn(x).$$

This function has two singularities at $x=0$ and 1. Now $f(x)$ needs to be expanded about these two points.

(a) Taylor's expansion about $x = 0$ is given by

$$\begin{aligned}
f(x) &= \ln|x|\, sgn(x)\left[1 + \frac{3}{2}|x| + O(|x|^2)\right] \\
&= \ln|x|\, sgn(x) + \frac{3}{2}|x| \ln|x|\, sgn(x) + O(|x|^2 \ln|x|\, sgn(x)) \\
&= F_1(x) + O(|x|^2 \ln|x|\, sgn(x)),
\end{aligned}$$

where

$$F_1(x) = \ln|x|\, sgn(x) + \frac{3}{2}|x| \ln|x|\, sgn(x).$$

(b) Taylor's expansion about $x=1$ is given by

$$\begin{aligned}
f(x) &= |1-x|^{-3/2}\left[\ln 1 + |x-1|(1) - \frac{1}{2}|x-1|^2 + O(|x-1|^3)\right] \\
&= |x-1|^{-1/2} - \frac{1}{2}|x-1|^{1/2} + O(|x-1|^{3/2}) \\
&= F_2(x) + O(|x-1|^{3/2}),
\end{aligned}$$

where

$$F_2(x) = |x-1|^{-1/2} - \frac{1}{2}|x-1|^{1/2}.$$

We know that if $\mathcal{F}\{f(x)\} = g(y)$, then $\mathcal{F}\{f(x \pm x_0)\} = e^{\pm(2\pi i y x_0)}\mathcal{F}\{f(x)\} = e^{\pm(2\pi i y x_0)}g(y)$. Let us consider that $G_1(y)$ and $G_2(y)$ are the Fourier transforms of $F_1(x)$ and $F_2(x)$, respectively, where these F's are the asymptotic expansions of $f(x)$ at the singularities, then by Theorem 4.4 with $N = 2$, we can write (like residue calculus)

$$g(y) = G_1(y) + G_2(y) + o(|y|^{-2}),$$

where

$$\begin{aligned} G_1(y) &= \mathcal{F}\{F_1(x)\} \\ &= \mathcal{F}\left\{\ln|x|\, sgn(x) + \frac{3}{2}|x|\ln|x|\, sgn(x)\right\} \\ &= \frac{i}{\pi y}[\ln(2\pi|y|) - \psi(0)] + \frac{3}{(2\pi y)^2}[\ln(2\pi|y|) - \psi(1)], \end{aligned}$$

$$\begin{aligned} G_2(y) &= \mathcal{F}\{F_2(x)\} \\ &= \mathcal{F}\left\{|x-1|^{-1/2} - \frac{1}{2}|x-1|^{1/2}\right\} \\ &= e^{-2\pi i y}\left[\mathcal{F}\left\{|x|^{-1/2} - \frac{1}{2}|x|^{1/2}\right\}\right] \\ &= e^{-2\pi i y}\left[-2\sqrt{\pi}(2\pi|y|)^{-1/2} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}\right]. \end{aligned}$$

Note that $\psi(n) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$. Hence $\psi(0) = -\gamma = -0.5772$. And $\psi(1) = -\gamma + 1$, where $\gamma = 0.5772$ is known as Euler's constant.

Thus writing the asymptotic result we have

$$\begin{aligned} g(y) &= G_1(y) + G_2(y) + o(|y|^{-2}) \\ &= \frac{i}{\pi y}[\ln(2\pi|y|) - \psi(0)] + \frac{3}{(2\pi y)^2}[\ln(2\pi|y|) - \psi(1)] \\ &\quad + e^{-2\pi i y}\left[-2\sqrt{\pi}(2\pi|y|)^{-1/2} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(2\pi|y|)^{-3/2}\right] + o(|y|^{-2}). \end{aligned}$$

We can explicitly write the Fourier transform of these orders as follows:

$$\begin{aligned}
\mathcal{F}\{O(x^2 \ln|x|\, sgn(x))\} &= O(\mathcal{F}\{(x^2 \ln|x|\, sgn(x))\}) \\
&= o\left[\frac{-4}{(2\pi i y)^3}(\ln(2\pi|y|) - \psi(2))\right] \\
&= o(|y|^{-3}), \\
\mathcal{F}\{O(|x-1|^{3/2}\} &= o[\mathcal{F}\{(|x-1|)^{3/2}\}] \\
&= o\left[-\frac{3\sqrt{\pi}}{2\sqrt{2}}e^{-2\pi i y}(2\pi|y|)^{-5/2}\right] \\
&= o(|y|^{-5/2}).
\end{aligned}$$

Remark

Note that

$$\begin{aligned}
\int_0^1 \frac{x^{-1/2}}{1+x}\,dx &= \frac{\pi}{2} + 0 = 1.5708, \\
\int_0^1 \frac{x^{-3/2}}{1+x}\,dx &= \frac{\pi}{2} + 1 = 2.5708, \\
\int_0^1 \frac{x^{-5/2}}{1+x}\,dx &= \frac{\pi}{2} + \frac{4}{3} = 2.9041.
\end{aligned}$$

4.5 Exercises

1. Prove that the Fourier transform of $f(x) = e^{ix^2}$ is $g(y) = e^{-i\pi^2 y^2}(1+i)\sqrt{\frac{\pi}{2}}$. [Hint: $\mathcal{F}\{f(x)\} = g(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x y}\,dx$.]
2. Determine the differences between the behaviour as $|y| \to \infty$ of the Fourier transforms of e^{-x^2} and $e^{-x^2} sgn(x)$.
3. Derive asymptotic development of $\int_1^2 e^{-2\pi i x y} \frac{\ln(2-x)}{(x-1)^{4/3}}\,dx$.
4. Obtain an asymptotic development of $\int_0^2 e^{-2\pi i x y} x \ln|x-1|\,dx$ with an error $o(|y|^{-3})$.
5. Find an asymptotic development of $\int_{-\infty}^{\infty} e^{-2\pi i y x^2} x^{-1} \ln|x-1|\,dx$ with an error $o(|y|^{-1})$.
6. Prove that

$$\begin{aligned}
\int_0^1 e^{-2\pi i y x^3}\,dx = {} & \frac{1}{3}!e^{-(\pi/6) i\, sgn(y)}|2\pi y|^{-1/3} \\
& -\frac{(-1/3)!}{(-4/3)!}e^{-2\pi i y + (\pi/2) i\, sgn(y)}|2\pi y|^{-1} + o(|y|^{\varepsilon - 2}),
\end{aligned}$$

where ε is arbitrarily small and positive.

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5 Fourier series as series of generalized functions

5.1 Introduction

This chapter deals with the study of convergence and uniqueness of trigonometric series as series of generalized functions. In its preparation we have used the following works of the authors, for example, Temple (1953, 1955), Lighthill (1964), Jones (1982), Champeney (1987) and Rahman (2001). We shall investigate how to determine the coefficients of a Fourier series. The chapter will deal with the existence of Fourier series representation for any periodic generalized function. Some study will be conducted about the asymptotic behaviour of Fourier coefficients. We shall demonstrate some examples of practical interest.

5.2 Convergence and uniqueness of a trigonometric series

We shall here state a theorem to manifest the convergence and uniqueness of trigonometric series (periodic series). We shall briefly prove that two different trigonometric series cannot converge to the same function.

Theorem 5.1

The trigonometric series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} \tag{5.1}$$

converges to a generalized function $f(x)$ if and only if $c_n = O(|n|^N)$ for some N as $|n| \to \infty$, in which case the Fourier transform of $f(x)$ is given by

$$g(y) = \sum_{n=-\infty}^{\infty} c_n \delta\left(y - \frac{n}{2\ell}\right). \tag{5.2}$$

The rigorous proof can be found in Lighthill's (1964) book. The series (5.1) converges to a generalized function $f(x)$ if and only if series (5.2), obtained by taking Fourier transforms term by term, converges to $g(y)$, the Fourier transform of $f(x)$. We can determine the Fourier transform of the Fourier series if we accept that the series converges. This is carried out as follows.

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}\right\} \\
&= \sum_{n=-\infty}^{\infty} \mathcal{F}\{c_n e^{in\pi x/\ell}\} \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} (e^{in\pi x/\ell}) \times (e^{-2\pi ixy})\, dx \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} e^{-2\pi i(y-n/2\ell)x}\, dx \\
&= \sum_{n=-\infty}^{\infty} c_n \delta\left(y - \frac{n}{2\ell}\right) = g(y).
\end{aligned}$$

Definition 1

A generalized function of the form (1.2) is called a "row of deltas" of spacing $\frac{1}{2\ell}$. A generalized function $f(x)$ is said to be periodic with period 2ℓ if $f(x) = f(x + 2\ell)$.

Example 1

Given that $f(x) = e^{in\pi x/\ell}$ is a periodic function with the fundamental period 2ℓ. Then by Theorem 5.1, its Fourier transform

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx = \delta\left(y - \frac{n}{2\ell}\right)$$

is also periodic. Theorem 5.1 states that the Fourier transform of a row of deltas of spacing $\frac{1}{2\ell}$ is a periodic function of period 2ℓ. This can be shown as follows:

$$\mathcal{F}\left\{\delta\left(x - \frac{n}{2\ell}\right)\right\} = \int_{-\infty}^{\infty} \delta\left(x - \frac{n}{2\ell}\right) e^{-2\pi ixy}\, dx = e^{-in\pi y/\ell}.$$

The inverse transform is

$$f(x) = \int_{-\infty}^{\infty} \delta\left(y - \frac{n}{2\ell}\right) e^{2\pi ixy}\, dx = e^{in\pi x/\ell}.$$

5.3 Determination of the coefficients in a trigonometric series

We shall now consider the problem of determining the coefficients in a trigonometric series.

If we know that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} \tag{5.3}$$

for some c_n, we shall explore a method to determine these coefficients. The classical solution of the present problem is

$$c_m = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-im\pi x/\ell}\, dx. \tag{5.4}$$

This solution is of no use where generalized functions are concerned, as these cannot be integrated between finite limits.

Example 2

No suitable definition of such integration is possible which will give a meaning to $\int_0^1 \delta'(x)\, dx = [\delta(x)|_0^1 = 0 - \infty]$. However, the idea of "integration over a period" can be reproduced by the use of a special kind of function.

Theorem 5.2

A "unitary function" $U(x)$ can be found, which is a good function vanishing for $|x| \geq 1$ and such that

$$\sum_{n=-\infty}^{\infty} U(x+n) = 1 \tag{5.5}$$

for all x. The Fourier transform $V(y)$ of any such function has $V(0) = 1$, but $V(m) = 0$ if m is an integer other than zero.

Proof

We can find many such functions $U(x)$ which is illustrated as follows. For any x, at most two terms of the series (5.5) differ from zero (those with $|x+n| < 1$). Therefore, it is necessary only that

$$U(x) + U(x-1) = 1 \quad \text{for } \; 0 \leq x \leq 1, \tag{5.6}$$

and that all derivatives of $U(x)$ should vanish at $x = \pm 1$ (so that they are continuous with zero, their value for $|x| > 1$). We may take

$$U(x) = \int_1^{|x|} \exp\left[-\frac{1}{t(1-t)}\right] dt \Big/ \int_0^1 \exp\left[-\frac{1}{t(1-t)}\right] dt, \tag{5.7}$$

for instance. The exponential ensures that all the derivatives of $U(x)$ vanish (with $U(x)$ itself) at $x=\pm 1$. Condition (5.6) is easily proved by making the substitution $t=1-s$ in the integral. The reader can easily verify that the *triangular function* defined below is a unitary function

$$U(x) = \begin{Bmatrix} 1-|x|, & -1 < x < 1 \\ 0, & |x| \geq 1 \end{Bmatrix}.$$

Lastly, if m is an integer,

$$\begin{aligned} V(m) &= \int_{-\infty}^{\infty} U(x) e^{-2\pi i m x}\, dx \\ &= \sum_{n=-\infty}^{\infty} \int_{n-1/2}^{n+1/2} e^{-2\pi i m x} U(x)\, dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-1/2}^{+1/2} e^{-2\pi i m x} U(x+n)\, dx \\ &= \int_{-1/2}^{1/2} e^{-2\pi i m x}\, dx \\ &= \begin{Bmatrix} 1 & (m=0) \\ 0 & (m\neq 0) \end{Bmatrix}, \end{aligned} \tag{5.8}$$

which completes the proof of Theorem 5.2.

We have demonstrated some numerical calculations of a few selected functions of $f(x)$ in the following Tables 5.1–5.4.

Table 5.1: Numerical values of the function $f(x) = \exp\left(-\frac{1}{x(1-x)}\right)$ in the range $0 \leq x \leq 1$.

x	$f(x) = \exp\left(-\frac{1}{x(1-x)}\right)$
0.0	0.000000
0.1	0.000015
0.2	0.001930
0.3	0.008549
0.4	0.015504
0.5	0.018316
0.6	0.015504
0.7	0.008549
0.8	0.001930
0.9	0.000015
1.0	0.000000

The idea of integrating a periodic function $f(x)$ over a period can now be replaced by the idea of integrating $f(x)U(x/2\ell)$ from $-\infty$ to ∞. For in this integral each value of $f(x)$ of the function is multiplied by just $\sum_{n=-\infty}^{\infty} U(x/2\ell + n) = 1$; but the integration is permissible in the theory of generalized functions since U is a good function.

Note that a unitary function in a complex field with parameter 2ℓ may be obtained by Fourier transforming a function λ in real field which has zero value outside of $(-1/2\ell, 1/2\ell)$ and has $\lambda(0) = 2\ell$. Thus $U(x)$ defined as follows for all x on $(-\infty, \infty)$ is a unitary function with parameter 2ℓ whenever $0 < 2\ell \leq 1$,

$$U(x) = \int_{-1}^{+1} \lambda(y) \exp(2\pi ixy)\, dy,$$

where

$$\lambda(y) = 2\ell \exp\left\{ \frac{-y^2}{1-y^2} \right\}, \quad -1 < y < 1.$$

A unitary function in a complex field is necessarily a unitary function in a real field, but not conversely.

Theorem 5.3

If $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$, then

$$c_m = \frac{1}{2\ell} \int_{-\infty}^{\infty} f(x)U(x/2\ell)e^{-im\pi x/\ell}\, dx, \tag{5.9}$$

where $U(x)$ is any unitary function.

Proof

We know that the right-hand side of eqn (5.9) can be written as

$$\begin{aligned}
&\frac{1}{2\ell} \int_{-\infty}^{\infty} f(x)U(x/2\ell)e^{-im\pi x/\ell}\, dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \int_{-\infty}^{\infty} U(x/2\ell)e^{i(n-m)\pi x}\, (dx/2\ell) \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n V(n-m) \\
&= c_m,
\end{aligned}$$

where $V(n-m)$ by Theorem 5.2 is 1 for $n = m$ and 0 for all other n.

5.4 Existence of Fourier series representation for any periodic generalized function

All the main objects of a theory of Fourier series listed in Section 1.4 have now been achieved, except that of proving that, if $f(x)$ is any periodic generalized function, and c_m are defined by eqn (5.9), then eqn (5.3) holds, or then $g(y)$, the Fourier transform of $f(x)$, satisfies $g(y) = \sum_{n=-\infty}^{\infty} c_n \delta(y - n/2\ell)$. The following theorem is a useful first step towards this, since $\sum_{n=-\infty}^{\infty} U(2\ell y - n) = 1$.

We shall state the following two theorems without proof. The interested reader should consult Lighthill's (1964) book for rigorous proofs.

Theorem 5.4

If $f(x)$ is a periodic generalized function with period 2ℓ and Fourier transform $g(y)$, and if

$$c_n = \frac{1}{2\ell}\int_{-\infty}^{\infty} f(x)U\left(\frac{x}{2\ell}\right)e^{-in\pi x/\ell}\,dx$$
$$= \int_{-\infty}^{\infty} g(y)V(n - 2\ell y)\,dy, \tag{5.10}$$

where the equality of the two forms of c_n follows from our previous Theorems 2.6 and 2.7, then

$$g(y)U(2\ell y - n) = c_n\delta(y - n/2\ell). \tag{5.11}$$

Theorem 5.5

If $a_{m,n}$ are such that $\sum_{n=0}^{\infty} x_n a_{m,n}$ is absolutely convergent and tends to a finite limit as $m \to \infty$ for any sequence x_n which is $O(n)$ as $n \to \infty$, then $\sum_{n=0}^{\infty} \lim_{m\to\infty} a_{m,n}$ converges to the sum $\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{m,n}$.

Theorem 5.6

If $g(y)$ is any generalized function, and $U(x)$ is a unitary function, then

$$g(y) = \sum_{n=-\infty}^{\infty} g(y)U(2\ell y - n). \tag{5.12}$$

Theorem 5.7

If $f(x)$ is any periodic generalized function with period 2ℓ and Fourier transform $g(y)$, then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$$
$$\text{and} \quad g(y) = \sum_{n=-\infty}^{\infty} c_n\delta(y - n/2\ell), \tag{5.13}$$

where

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\,dx$$
$$= \frac{1}{2\ell}\int_{-\infty}^{\infty} f(x)U(x/2\ell)e^{-in\pi x/\ell}\,dx. \tag{5.14}$$

Proof

We know that

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx\\ &= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}\right)e^{-2\pi ixy}\,dx\\ &= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} e^{-2\pi i(y-n/2\ell)}\,dx\\ &= \sum_{n=-\infty}^{\infty} c_n\delta(y-n/2\ell)\\ &= g(y),\end{aligned}$$

where $c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\,dx$.

It is worth noting that if in addition, $f(x)$ is absolutely integrable (as assumed in the classical theory of Fourier series), then

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\,dx, \tag{5.15}$$

since for such an $f(x)$ we can write

$$\begin{aligned}c_n &= \frac{1}{2\ell}\sum_{m=-\infty}^{\infty}\int_{(2m-1)\ell}^{(2m+1)\ell} f(x)U(x/2\ell)e^{-in\pi x/\ell}\,dx\\ &= \frac{1}{2\ell}\sum_{m=-\infty}^{\infty}\int_{-\ell}^{\ell} f(x)U(x/2\ell+m)e^{-in\pi x/\ell}\,dx\\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\left\{\sum_{m=-\infty}^{\infty} U(x/2\ell+m)\right\}e^{-in\pi x/\ell}\,dx\\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\,dx.\end{aligned}$$

Therefore the classical Fourier series theory is included in our more general result.

Theorem 5.8

Under the conditions of Theorem 5.7, c_n is necessarily $O(|n|^N)$ for some N as $|n| \to \infty$. Hence

$$f(x) = \frac{d^{N+2} f_1(x)}{dx^{N+2}},$$
$$\text{where } f_1(x) = c_0 \frac{x^{N+2}}{(N+2)!} + \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty}\right) c_n \left(\frac{1}{in\pi}\right)^{N+2} e^{in\pi x/\ell} \tag{5.16}$$

is a continuous function.

Proof

By Theorem 5.1, the series (5.13) which have just been proved to converge could not do so unless $c_n = O(|n|^N)$ for some N. The first of eqn (5.16) is obtained by term-by-term differentiation. The fact that $f_1(x)$ is continuous follows from the fact that the series for it is absolutely and uniformly convergent, by comparison with the series $\sum_{n=1}^{\infty} n^{-2}$.

The fact that a periodic generalized function must necessarily be a repeated derivative of some continuous function is interesting as it shows that there is a limit to the seriousness of the singularities that these functions can have.

We have now derived all the general properties which are necessary for a satisfactory theory of Fourier series; there exists a unique Fourier series representation of any periodic function, which converges to the function, whose coefficients can be determined and which can be differentiated term by term. In the following we shall illustrate some specific problems.

5.5 Some practical examples: Poisson's summation formula

This section is mainly concerned with the consequences of the following result.

Example 3

Determine the Fourier coefficients of the following periodic generalized function $f(x) = \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell)$. Then find the Fourier transform of this periodic generalized function.

Solution

The generalized function

$$f(x) = \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell) \tag{5.17}$$

exists by Theorem 5.1 and obviously is periodic with period 2ℓ. Its nth Fourier coefficient is

$$\begin{aligned} c_n &= \frac{1}{2\ell}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(x-2m\ell)U\left(\frac{x}{2\ell}\right)e^{-in\pi x/\ell}\,dx \\ &= \frac{1}{2\ell}\sum_{m=-\infty}^{\infty}U(m) \\ &= \frac{1}{2\ell}, \end{aligned} \tag{5.18}$$

and so by Theorem 5.7

$$f(x) = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}e^{in\pi x/\ell}, \tag{5.19}$$

an equation whose real form

$$\sum_{m=-\infty}^{\infty}\delta(x-2m\ell) = \frac{1}{2\ell} + \frac{1}{\ell}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell} \tag{5.20}$$

is worth noting; and the Fourier transform of $f(x)$ is

$$g(y) = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}\delta\left(y-\frac{n}{2\ell}\right). \tag{5.21}$$

In words, a row of equal deltas has as its Fourier transform a row of equal deltas. This is an interesting result.

Example 4

By differentiation or otherwise, evaluate $\sum_{m=-\infty}^{\infty}\delta'(x-2m\ell)$.

Solution

From eqn (5.20), differentiating with respect to x, we obtain

$$\sum_{m=-\infty}^{\infty}\delta'(x-2m\ell) = -\frac{\pi}{\ell^2}\sum_{n=1}^{\infty}n\sin\frac{n\pi x}{\ell}. \tag{5.22}$$

Remark

This is a trigonometric series in which the old-fashioned "summability" methods of handling divergent series which preceded the introduction of generalized functions broke down. The generalized functions gave the sum of the trignometric series as zero for $x \neq 2m\ell$ (which is correct, as the generalized function is equal to 0 in any interval not including those points), but also gave it as zero for $x = 2m\ell$, on the ground that every term of the series vanished at these points. Thus they missed the true character of the singularity at $x = 2m\ell$; and what was perhaps even worse, uniqueness was absent in these theories, because of the existence of trigonometrical series whose sum was everywhere zero.

Example 5

Let us consider the periodic function $f(x) = x$ defined in the range $(-\ell, \ell)$. Obtain its Fourier series representation as an ordinary function. Then by differentiating the series term by term, determine the series representation of 1. Discuss the drawback.

Solution

In the classical theory of Fourier series we calculate the nth Fourier coefficient of the function $f(x) = x$ in the range $(-\ell, \ell)$ as

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} x e^{-in\pi x/\ell}\, dx = \frac{i\ell}{n\pi}(-1)^n, \tag{5.23}$$

which gives

$$f(x) = x = \frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{\ell} \tag{5.24}$$

as the full-range Fourier series of x in $(-\ell, \ell)$. Now, it is usually stated that such a series cannot be differentiated term by term; and, in fact, the Fourier series of 1 in $(-\ell, \ell)$ is not

$$2\sum_{n=1}^{\infty} (-1)^{n-1} \cos\frac{n\pi x}{\ell}$$

but

$$1 + \sum_{n=1}^{\infty} (0) \cos\frac{n\pi x}{\ell}.$$

However, in the theory of generalized functions, we have seen that any series can be differentiated term by term. To reconcile the apparent contradiction, note that the sum of eqn (5.24) is not x, but rather a periodic function which coincides with x in the period $(-\ell, \ell)$. Thus, if

$$f(x) = \begin{cases} x & (-\ell, \ell) \\ x - 2m\ell & [(2m-1)\ell < x < (2m+1)\ell], \end{cases} \tag{5.25}$$

then by Theorem 5.7 and by differentiation of eqn (5.24) with respect to x, we obtain

$$f'(x) = 1 = 2\sum_{n=1}^{\infty}(-1)^{n-1}\cos\frac{n\pi x}{\ell}. \tag{5.26}$$

But by eqn (5.25)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x - 2\ell\sum_{m=1}^{\infty}H(x-(2m-1)\ell) + 2\ell\sum_{m=-\infty}^{0}H((2m-1)\ell - x)\right] \\ &= 1 - 2\ell\sum_{m=-\infty}^{\infty}\delta(x-(2m-1)\ell). \end{aligned} \tag{5.27}$$

This $f'(x)$ is not the function 1. In fact, by eqn (5.20), it is

$$-2\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi x}{\ell} \tag{5.28}$$

in perfect agreement with eqn (5.26).

We have learned something useful by applying Parseval's formula to the result of Example 3.

Theorem 5.9: Poisson's summation formula

If $F(x)$ is a good function and $G(y)$ its Fourier transform, then

$$\sum_{m=-\infty}^{\infty}F(\lambda m) = \frac{1}{\lambda}\sum_{n=-\infty}^{\infty}G\left(\frac{n}{\lambda}\right). \tag{5.29}$$

Proof

We know that

$$\begin{aligned} g(y) &= \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx, \\ f(x) &= \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy, \\ G(y) &= \mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x)e^{-2\pi ixy}\,dx, \\ F(x) &= \int_{-\infty}^{\infty} G(y)e^{2\pi ixy}\,dy. \end{aligned}$$

Using these information, we at once find that

$$\int_{-\infty}^{\infty} f(x)F(-x)\,dx = \int_{-\infty}^{\infty} g(y)G(y)\,dy.$$

Thus if we consider that

$$f(x) = \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell) = \sum_{m=-\infty}^{\infty} \delta(x - \lambda m),$$

then its Fourier transform can be written as

$$g(y) = \frac{1}{2\ell}\sum_{-\infty}^{\infty} \delta\left(y - \frac{n}{2\ell}\right) = \frac{1}{\lambda}\sum_{-\infty}^{\infty} \delta\left(y - \frac{n}{\lambda}\right),$$

where $\lambda = 2\ell$.

Thus we have

$$\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - \lambda m)F(-x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \delta\left(y - \frac{n}{\lambda}\right)G(y)\,dy$$

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - \lambda m)F(-x)\,dx = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} \int_{-\infty}^{\infty} \delta\left(y - \frac{n}{\lambda}\right)G(y)\,dy$$

$$\sum_{m=-\infty}^{\infty} F(\lambda m) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} G\left(\frac{n}{\lambda}\right).$$

Note that for even good function $F(-\lambda m) = F(\lambda m)$.

Example 6

If $F(x) = e^{-x^2}$, then $G(y) = \sqrt{\pi}e^{-\pi^2 y^2}$. And hence show that

$$\sum_{m=-\infty}^{\infty} e^{-m^2\lambda^2} = \frac{\sqrt{\pi}}{\lambda} \sum_{n=-\infty}^{\infty} e^{-n^2\pi^2/\lambda^2}.$$

Proof

Using the above result yields

$$\sum_{m=-\infty}^{\infty} e^{-m^2\lambda^2} = \frac{\sqrt{\pi}}{\lambda} \sum_{n=-\infty}^{\infty} e^{-n^2\pi^2/\lambda^2}.$$

It is obvious that this relationship holds good only for $\lambda = \sqrt{\pi}$. The left-hand side converges very rapidly for $\lambda \geq \sqrt{\pi}$, and so does the right-hand side for $\lambda < \sqrt{\pi}$ – in both cases, faster than

$$1 + 2e^{-\pi} + 2e^{-4\pi} + 2e^{-8\pi} + \cdots = 1 + 0.0864278 + 0.0000070 + (10^{-12}) + \cdots .$$

Example 7

If $F(x) = e^{-x^2 - 2\pi ixz}$, then its Fourier transform is $G(y) = \sqrt{\pi}e^{-\pi^2(y+z)^2}$. Then by Theorem 5.9 (Poisson's summation formula) show that

$$1 + 2\sum_{m=1}^{\infty} e^{-m^2\lambda^2}\cos(2\pi m\lambda z) = \frac{\sqrt{\pi}}{\lambda}e^{-\pi^2 z^2}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2/\lambda^2}\cosh\left(\frac{2\pi^2 nz}{\lambda}\right)\right],$$

which is Jacobi's transformation in the theory of theta function.

Proof

$$\begin{aligned}
\mathcal{F}\{F(x)\} &= \int_{-\infty}^{\infty} (e^{-x^2 - 2\pi ixz})e^{-2\pi ixy}\,dx \\
&= e^{-\pi^2(y+z)^2}\int_{-\infty}^{\infty} e^{-(x+\pi i(y+z))^2}\,dx \\
&= \sqrt{\pi}e^{-\pi^2(y+z)^2} \\
&= G(y).
\end{aligned}$$

Hence by Poisson's summation formula, we have

$$\sum_{m=-\infty}^{\infty} F(\lambda m) = \frac{1}{\lambda}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{\lambda}\right)$$

$$\sum_{m=-\infty}^{\infty} e^{-\lambda^2 m^2 - 2\pi i\lambda mz} = \frac{\sqrt{\pi}}{\lambda}\sum_{n=-\infty}^{\infty} e^{-\pi^2(n/\lambda + z)^2}$$

$$1 + 2\sum_{m=-\infty}^{\infty} e^{-\lambda^2 m^2}\cos(2\pi\lambda mz) = \frac{\sqrt{\pi}}{\lambda}e^{-\pi^2 z^2}\left[1 + 2\sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/\lambda^2}\cosh\frac{2\pi^2 nz}{\lambda}\right],$$

which is Jacobi's transformation in the theory of theta functions.

Remark

Similar remarks about convergence apply. The cosh in the series on the right-hand side of the above series worsens its convergence only slightly, since the above expression is a periodic function of z with period λ^{-1} and so one may take $|z| < \frac{1}{2}\lambda^{-1}$ for computational purposes. And the equation is therefore the key to the computation of the theta functions and, through them, of the elliptic functions.

Example 8

In the period $0 < z < 2\pi$, show that

$$1 + 2\sum_{n=1}^{\infty} \frac{\cos nz}{1+n^2} = \frac{\pi \cosh(\pi - z)}{\sinh \pi}.$$

Proof

We can write the left-hand side as

$$1 + 2\sum_{n=1}^{\infty} \frac{\cos nz}{1+n^2} = \sum_{m=-\infty}^{\infty} \frac{e^{imz}}{1+m^2} = \sum_{m=-\infty}^{\infty} F(m),$$

where $F(x) = \frac{e^{izx}}{1+x^2}$. The Fourier transform of $F(x)$ is given by

$$\begin{aligned}
\mathcal{F}\{F(x)\} &= \int_{-\infty}^{\infty} \left(\frac{e^{izx}}{1+x^2}\right) e^{-2\pi ixy}\, dx \\
&= \int_{-\infty}^{\infty} \frac{e^{-i(2\pi y - z)x}}{1+x^2}\, dx \\
&= 2\pi i \times \text{sum of the residues at the poles } x = \pm i \\
&= \frac{2\pi i}{2i} e^{-|2\pi y - z|} \\
&= \pi e^{-|2\pi y - z|}.
\end{aligned}$$

Here we used the semi-circular contour on the upper half of the circle of infinite radius.

Hence $G(y) = \pi e^{-|2\pi y - z|}$. By Poisson's summation formula we have

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} F(m) &= \sum_{n=-\infty}^{\infty} G(n) \\
&= \sum_{n=-\infty}^{0} \pi e^{-(z-2\pi n)} + \sum_{n=1}^{\infty} \pi e^{-(2\pi n - z)} \\
&= \frac{\pi[e^{-z} + e^{z-2\pi}]}{1 - e^{-2\pi}} \\
&= \frac{\pi \cosh(\pi - z)}{\sinh \pi}.
\end{aligned}$$

5.6 Asymptotic behaviour of the coefficients in a Fourier series

The following theorem enables us to use the method of Chapter 4 to find the asymptotic behaviour as $|n| \to \infty$ for the Fourier coefficients c_n of a given function $f(x)$.

Theorem 5.10

If $f(x)$ is a periodic generalized function with period 2ℓ, then $C(y)$, the Fourier transform of $(2\ell)^{-1}f(x)U(x/2\ell)$, is a continuous function whose value for $y = n/2\ell$ is the nth Fourier coefficient c_n of $f(x)$.

Theorem 5.11

If $f(x)$ is a periodic generalized function, with a finite number of singularities $x = x_1, x_2, \ldots, x_M$ in the period $-\ell < x \leq \ell$, such that for each m from 1 to M, $f(x) - F_m(x)$ has absolutely integrable Nth derivative in an interval including x_m, where $F_m(x)$ is a linear combination of functions of the type

$$\begin{aligned} &|x - x_m|^\beta, \\ &|x - x_m|^\beta\, sgn(x - x_m), \\ &|x - x_m|^\beta \ln|x - x_m|, \\ &|x - x_m|^\beta \ln|x - x_m|\, sgn(x - x_m) \\ &\text{and} \;\; \delta^{(p)}(x - x_m) \end{aligned} \tag{5.30}$$

for different values of β and p, then c_n, the nth Fourier coefficient of $f(x)$, satisfies

$$c_n = \frac{1}{2\ell}\sum_{m=1}^{M} G_m\left(\frac{n}{2\ell}\right) + o(|n|^{-N}) \quad \text{as } |n| \to \infty, \tag{5.31}$$

where $G_m(y)$, the Fourier transform of $F_m(x)$, can be obtained from Formulae 1–18 (Table 4.2).

Example 9

Find an asymptotic expression, with error $o(|n|^{-9})$, for the nth Fourier coefficient c_n of the periodic function

$$f(x) = e^{|\cos x|^3}.$$

Solution

The period of the given function is 2π and the interval is $-\pi < x < \pi$. The singularities of $f(x)$ are when $\cos x = 0$, that is $x = \pm\pi/2$. When $x \to \pm\pi/2$,

$$f(x) = 1 + \frac{|\cos x|^3}{1!} + \frac{|\cos x|^6}{2!} + \frac{|\cos x|^9}{3!} + \cdots,$$

where

$$|\cos x| = |\sin(x \mp \pi/2)| = |x \mp \pi/2| - \frac{1}{3!}|x \mp \pi/2|^3 + \frac{1}{5!}|x \mp \pi/2|^5 - \cdots.$$

Hence inserting these information, $f(x)$ can be expanded as

$$f(x) = 1 + |x \mp \pi/2|^3 - \frac{1}{2}|x \mp \pi/2|^5 + \frac{13}{120}|x \mp \pi/2|^7 + \frac{1}{2}|x \mp \pi/2|^6 - \frac{1}{2}|x \mp \pi/2|^8 + O(|x \mp \pi/2|^9)$$

as $x \to \mp\pi/2$. Hence, by Theorem 5.11 with $N = 9$ and $\ell = \pi$, and Table 4.2,

$$c_n = \frac{1}{2\pi}(e^{-\frac{1}{2}n\pi i} + e^{\frac{1}{2}n\pi i})\left[\frac{2(3!)}{(in)^4} - \frac{1}{2}\frac{2(5!)}{(in)^6} + \frac{13}{120}\frac{2(7!)}{(in)^8} + o(|n|^{-9})\right]$$
$$= \frac{12}{\pi}\cos\left(\frac{1}{2}n\pi\right)\left(\frac{1}{n^4} + \frac{10}{n^6} + \frac{91}{n^8} + O\left(\frac{1}{n^{10}}\right)\right),$$

where the precise form of the error term follows from the detailed expression for error in the expression of $f(x)$ presented above. Note that in evaluating the Fourier transforms we have used Formula 5 in Table 4.2. That means, for example,

$$\mathcal{F}\{|x \mp \pi/2|^3\} = e^{\mp in\pi/2}\mathcal{F}\{|x|^3\} = e^{\mp in\pi/2}\mathcal{F}\{x^3\, sgn(x)\} = e^{\mp in\pi/2}\frac{2(3!)}{(2\pi iy)^4}.$$

Example 10

Find an asymptotic expression for the coefficients b_n in the Fourier sine series for $f(x) = \sqrt{x}$ in the range $0 < x < \ell$, with an error $o(n^{-2})$ as $n \to \infty$.

Solution

The Fourier sine series is simply the full-range Fourier series of the odd function $f(x) = |x|^{1/2}\, sgn(x)$ in $(-\ell, \ell)$; and the coefficients b_n in the sine series are $2i$ times the Fourier coefficient c_n of the periodic function $f(x)$ which equals $|x|^{1/2}\, sgn(x)$ in the period $-\ell < x \leq \ell$. Now $f(x)$ has singularities at $x = 0$ and ℓ in this period, and

$$f(x) = |x|^{1/2}\, sgn(x),$$
$$f(x) = -\ell^{1/2}\, sgn(x-\ell) + \frac{1}{2}\ell^{-1/2}(x-\ell) + O(|x-\ell|^2),$$

as $x \to 0$ (where for once the error in the expression quoted is zero!) and as $x \to \ell$. Hence by Theorem 5.11 with $N = 2$, and Table 4.2,

$$b_n = 2ic_n = \frac{i}{\ell}\left(\frac{-i\sqrt{\pi/2}}{(\pi n)^{3/2}} - \ell^{1/2} e^{-n\pi i}\frac{2\ell}{n\pi i} + o(n^{-2})\right)$$
$$= \frac{2(-1)^{n-1}}{n\pi}\sqrt{\ell} + \frac{\sqrt{\ell/2}}{\pi n^{3/2}} + O(n^{-3}),$$

as $n \to \infty$, where the precise form of the error term follows from the detailed expression for the error in the above expression for $f(x)$.

Note that we have used the following formula to obtain the coefficients c_n:

$$c_n = \frac{1}{2\ell}\sum_{m=1}^{M} G_m\left(\frac{n}{2\ell}\right) + o(|n|^{-N}) \quad \text{as } |n| \to \infty, \tag{5.32}$$

where $G_m(y)$, the Fourier transform of $F_m(x)$, can be obtained from Formulae 1–18 (Table 4.2).

Remark

This example exhibits a common feature, where Fourier series representation of continuous functions in a limited range is concerned – namely, that the worst singularity of that periodic function, which coincides with the given function in the range, is a simple discontinuity at one of the end-points. By Theorem 5.11 and Table 4.2, the nth Fourier coefficient for large n is then necessarily of order n^{-1}. But there are, of course, exceptions to this rule, that is, functions for which no such discontinuity occurs at either end-point.

Example 11

Sum the series $\sum_{n=1}^{\infty} \frac{n \sin nz}{1+n^4}$ for $0 < z < 2\pi$.

Solution

We can write the given series as

$$\sum_{n=1}^{\infty} \frac{n \sin nz}{1+n^4} = \sum_{m=-\infty}^{\infty} \frac{1}{2i}\frac{e^{imz}}{1+m^4} = \sum_{m=-\infty}^{\infty} F(m),$$

where $F(x) = \frac{1}{2i}\frac{xe^{izx}}{1+x^4}$. The Fourier transform of $F(x)$ is given by

$$\begin{aligned}
\mathcal{F}\{F(x)\} &= \frac{1}{2i}\int_{-\infty}^{\infty}\left(\frac{xe^{izx}}{1+x^4}\right)e^{-2\pi ixy}\,dx \\
&= \frac{1}{2i}\int_{-\infty}^{\infty}\frac{xe^{-i(2\pi y - z)x}}{1+x^4}dx \\
&= (2\pi i)\left(\frac{1}{2i}\right)\times \text{sum of the residues} \\
&\quad \text{at the poles } x = e^{\pi i/4} \text{ and } x = e^{3\pi i/4} \\
&= \pi e^{-\sqrt{2}|2\pi y - z|}\frac{1}{4i}[e^{i\sqrt{2}(2\pi y - z)} - e^{-i\sqrt{2}(2\pi y - z)}] \\
&= \frac{\pi}{2}e^{-\sqrt{2}|2\pi y - z|}\sin(\sqrt{2}(2\pi y - z)).
\end{aligned}$$

Here we used the semi-circular contour on the upper half of the circle of infinite radius.

Hence $G(y) = \frac{\pi}{2}e^{-\sqrt{2}|2\pi y - z|}\sin(\sqrt{2}(2\pi y - z))$. By Poisson's summation formula we have

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} F(m) &= \sum_{n=-\infty}^{\infty} G(n) \\
&= \sum_{n=-\infty}^{\infty}\frac{\pi}{2}e^{-\sqrt{2}|2\pi n - z|}\sin(\sqrt{2}(2\pi n - z))
\end{aligned}$$

Example 12

Find an asymptotic expansion for the nth Fourier coefficient c_n of the periodic function $f(x) = |1 + 2\sin x|^{-1/3}$, with an error $o(|n|^{-2})$, and state the precise order of magnitude of the error.

Solution

The period of the given function is 2π and the interval is $0 < x < 2\pi$. The singularities of $f(x)$ are when $1 + 2\sin x = 0$, that is $x = 7\pi/6$ and $11\pi/6$. By binomial expansion we have

$$f(x) = |1 + 2\sin x|^{-1/3} = 1 - \frac{2}{3}|\sin x| + \frac{8}{9}|\sin x|^2 - \frac{112}{81}|\sin x|^3 + \cdots,$$

where $x \to 7\pi/6$

$$\begin{aligned}
|\sin x| = |\sin(x + 7\pi/6)| &= |\sin(x + \pi/6)| \\
&= |x + \pi/6| - \frac{1}{3!}|x + \pi/6|^3 + \frac{1}{5!}|x + \pi/6|^5 - \cdots,
\end{aligned}$$

Table 5.2: Numerical values of the periodic function $f(x) = |1 + 2\sin x|$ for one period $0 \le x \le 2\pi$.

x	$f(x) = \|1 + 2\sin x\|$
0.0	1.000
$\pi/6$	2.000
$2\pi/6$	2.732
$3\pi/6$	3.000
$4\pi/6$	2.732
$5\pi/6$	2.000
$6\pi/6$	1.000
$7\pi/6$	0.000
$8\pi/6$	0.732
$9\pi/6$	1.000
$10\pi/6$	0.732
$11\pi/6$	0.000
$12\pi/6$	1.000

Table 5.3: Numerical values of the periodic function $f(-x) = |1 - 2\sin x|$ for one period $-2\pi \le -x \le 0$.

$-x$	$f(-x) = \|1 - 2\sin x\|$
0.0	1.000
$-\pi/6$	2.000
$-2\pi/6$	2.732
$-3\pi/6$	3.000
$-4\pi/6$	2.732
$-5\pi/6$	2.000
$-6\pi/6$	1.000
$-7\pi/6$	0.000
$-8\pi/6$	0.732
$-9\pi/6$	1.000
$-10\pi/6$	0.732
$-11\pi/6$	0.000
$-12\pi/6$	1.000

where $x \to 11\pi/6$

$$|\sin x| = |\sin(x + 11\pi/6)| = |\sin(x - \pi/6)| \\ = |x - \pi/6| - \frac{1}{3!}|x - \pi/6|^3 + \frac{1}{5!}|x - \pi/6|^5 - \cdots .$$

Hence inserting these information, $f(x)$ can be expanded as

$$f(x) = 1 - \frac{2}{3}|x \pm \pi/6| + \frac{17}{9}|x \pm \pi/6|^2 + \frac{1}{9}|x \pm \pi/6|^3 + O(|x \pm \pi/6|^4)$$

as $x \to \pm\pi/6$. Hence, by Theorem 5.11 with $N = 9$ and $\ell = \pi$, and Table 4.2

$$\begin{aligned} c_n &= -\frac{1}{2\pi}(e^{+\frac{1}{6}n\pi i} + e^{-\frac{1}{6}n\pi i})\left[\frac{2}{3}\frac{2(1!)}{(in)^2} - \frac{17}{9}\frac{2(2!)}{(in)^3} - \frac{1}{9}\frac{2(3!)}{(in)^4} + o(|n|^{-5})\right] \\ &= -\frac{1}{\pi}\cos\left(\frac{1}{6}n\pi\right)\left(-\frac{4}{3n^2} - \frac{4}{3n^4} + O\left(\frac{1}{n^5}\right)\right) \\ &= \frac{4}{3\pi}\cos\left(\frac{1}{6}n\pi\right)\left(\frac{1}{n^2} + \frac{1}{n^4} + O\left(\frac{1}{n^5}\right)\right), \end{aligned}$$

where the precise form of the error term follows from the detailed expression for error in the expression of $f(x)$ presented above. Note that in evaluating the Fourier transforms we have used Formula 5 in Table 4.2. That means, for example,

$$\mathcal{F}\{|x \pm \pi/6|^3\} = e^{\pm in\pi/6}\mathcal{F}\{|x|^3\} = e^{\pm in\pi/6}\mathcal{F}\{x^3\, sgn(x)\} = e^{\pm in\pi/6}\frac{2(3!)}{(2\pi iy)^4}.$$

Example 13

Find an asymptotic expansion for the coefficients a_n in the Fourier cosine series for $f(x) = x\ln x$ in the range $0 < x < 1$, with an error $o(n^{-3})$, and state the precise order of magnitude of the error.

Solution

The Fourier cosine series is simply the full-range Fourier cosine of the even function $f(x) = x\ln x$ in $(-1, 1)$; and the coefficients a_n in the cosine series are 2 times the Fourier coefficient c_n of the periodic function $f(x)$ which equals $x\ln x$ in the period $-1 < x \le 1$. Now $f(x)$ has singularities at $x = 0$ and 1 in this period, and

$$\begin{aligned} f(x) &= |x\ln x|, \\ f(x) &= |x - 1| + \frac{1}{2}|x - 1|^2 + O(|x - 1|^3), \end{aligned}$$

Table 5.4: Numerical values of the periodic function $f(x) = x \ln x$ in the range $0 \le x \le 1$.

x	$f(x) = x \ln x$
0.0	0.0000
0.1	−0.2303
0.2	−0.3219
0.3	−0.3612
0.4	−0.3665
0.5	−0.3466
0.6	−0.3065
0.7	−0.2497
0.8	−0.1785
0.9	−0.0948
1.0	−0.0000
1.1	0.1048
1.2	0.2188
1.3	0.3411
1.4	0.4711
1.5	0.6082

as $x \to 0$ and as $x \to 1$. Hence by Theorem 5.11 with $N = 2$, and Table 4.2,

$$\begin{aligned} a_n = 2c_n &= \frac{2}{2}\left(\frac{\pi i}{(\pi^2 n)^2} - \frac{2e^{-in\pi}}{n^2\pi^2} + o(n^{-3})\right) \\ &= \frac{1}{n^2\pi^2}[\pi i - 2(-1)^n + O(n^{-3})], \end{aligned}$$

as $n \to \infty$, where the precise form of the error term follows from the detailed expression for the error in the above expression of $f(x)$.

Note that we have used the following formula in obtaining the coefficients c_n:

$$c_n = \frac{1}{2\ell}\sum_{m=1}^{M} G_m\left(\frac{n}{2\ell}\right) + o(|n|^{-N}) \quad \text{as } |n| \to \infty, \tag{5.33}$$

where $G_m(y)$, the Fourier transform of $F_m(x)$, can be obtained from Formulae 1–18 (Table 4.2).

Remark

This example exhibits a common feature, where Fourier series representation of continuous functions in a limited range is concerned – namely, that the worst singularity of that periodic function, which coincides with the given function in the range, is a simple discontinuity at one of the end-points. By Theorem 5.11 and Table 4.2, the nth Fourier coefficient for large n is then necessarily of order n^{-1}. But there are, of course, exceptions to this rule, that is, functions for which no such discontinuity occurs at either end-point.

5.7 Exercises

1. Find the Fourier series of the generalized function with period 2π which equals e^x in $(-\pi, \pi)$. Deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{1}{2}(\pi \operatorname{cosech} \pi + 1).$$

2. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(-i\alpha\cos x - i\alpha x)\,dx = \sum_{n=-\infty}^{\infty} \exp(-in\pi/\ell)\,J_n(\alpha)\delta(\alpha - n).$$

[Hint: $\frac{e^{in\pi/2}}{2\pi}\int_0^{2\pi} e^{-i\alpha\cos x - inx}\,dx = J_n(\alpha)$.]

3. Show that

$$\sum_{n=1}^{\infty} e^{inx} = \pi \sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) + \frac{1}{2} i \cot\frac{x}{2} - \frac{1}{2}.$$

4. By considering the good function $\exp(-x^2 - 2\pi iyx)$ prove that

$$1+2\sum_{m=1}^{\infty} e^{-m^2\lambda^2}\cos(2m\pi\lambda y) = \frac{\sqrt{\pi}}{\lambda}e^{-\pi^2 y^2}\left[1+2\sum_{n=1}^{\infty} e^{-n^2\pi^2/\lambda^2}\cosh\frac{2\pi^2 ny}{\lambda}\right].$$

If $y = \frac{\lambda}{4}$ estimate the sum when (a) $\lambda = 1$, (b) $\lambda = 10$.

5. If $f(x)$ is continuous and of bounded variation on $0 \le x < \infty$, tends to 0 as $x \to \infty$ and is such that $\int_0^\infty f(x)\,dx$ converges, prove that

$$\frac{1}{2}f(0) + \sum_{m=1}^{\infty} f(m\lambda) = \frac{1}{\lambda}\left\{\frac{1}{2}g(0) + \sum_{n=1}^{\infty} g\left(\frac{2\pi n}{\lambda}\right)\right\},$$

where $g(y) = 2\int_0^\infty f(x)\cos xy\,dx$.

6. If $f(x)$ is an even good function prove that

$$\frac{1}{2}f(0)+\sum_{m=1}^{\infty}f(m\lambda)=\frac{1}{\lambda}\left\{\frac{1}{2}g(0)+\sum_{n=1}^{\infty}g\left(\frac{2\pi n}{\lambda}\right)\right\},$$

where $g(y)=2\int_0^{\infty}f(x)\cos xy\,dx$.

Remark

The Fourier transforms can be defined in two ways. The first definition is

$$\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty}f(x)e^{-2\pi ixy}\,dx=g(y),$$
$$f(x)=\int_{-\infty}^{\infty}g(y)e^{2\pi ixy}\,dy=\mathcal{F}^{-1}\{g(y)\}.$$

The second definition (conventional) is

$$\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty}f(x)e^{-ikx}\,dx=g(k),$$
$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(k)e^{ikx}\,dk=\mathcal{F}^{-1}\{g(k)\}.$$

They are equivalent. It is obvious that $2\pi y=k$ such that $dy=\frac{1}{2\pi}dk$.

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6 The fast Fourier transform (FFT)

6.1 Introduction

In the last five chapters we have discussed the application of Fourier transforms to a number of generalized functions. We have derived some important mathematical formulations and illustrated our discussion with some practical examples. In this chapter, we shall introduce some ideas and concepts regarding how the Fourier transform plays a vital role in our daily lives. This brings me to examine from theoretical and mathematical viewpoints an important topic called the *fast Fourier transform* (FFT). The FFT is a mathematical algorithm which is an extremely important and widely used method that is used to extract information from sampled signals. In this twenty-first century, which is an age of computers, electronics dominate our life style tremendously. We can consider the FFT simply as a tool without getting too involved in the mathematical aspects, but since it is a book which is meant for the mathematically inclined reader, it would be inappropriate if we do not give some mathematical basis about the FFT. In short we can say that the Fourier transform is a mathematical procedure which can be thought of as transforming a function from the time domain to the frequency domain. The application of the Fourier transform to a signal is analogous to the splitting up of a light beam by a prism to form the optical *spectrum* of the light source. An optical spectrum consists of lines or bands of colour corresponding to the various wavelengths and hence different frequencies of light wave emitted by the source. The spectrum of a signal in a digital signal processing refers to the way energy in the signal is distributed over its various frequency components.

Digital signal processing involves *discrete* signals, that is, signals which are sampled at regular intervals of time rather than continuous signals. A modified form of the Fourier transform, known as the *discrete Fourier transform* (DFT), is used in the case of sampled (discrete) signals. To compute the DFT of a signal comprising 1000 samples, say, would entail of the order of one million (1000^2) calculations. The DFT is therefore an extremely numerically intensive procedure. With this procedure we get extremely accurate information about the frequency components of a signal with the huge computational effort involved. With the development of the digital computer it is not a problem to perform numerical calculation rapidly and accurately. These calculations were performed by using DFT procedure until the 1960s when Cooley & Tukey (1965) discovered a numerical algorithm which allows the DFT to be evaluated with a significant reduction in the amount of calculation required. This algorithm, called the fast Fourier transform, or FFT, allows the DFT of a sampled signal to be obtained rapidly and efficiently. Nowadays, the FFT is used in many areas of applied problems because of its rapidity, accuracy and efficiency. Thus the FFT is nothing but the DFT which is very popular among scientists and engineers because of the reason cited above.

6.2 Some preliminaries leading to the fast Fourier transforms

We have already developed the formula for the Fourier transform and its inverse which are given by

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx = g(y), \tag{6.1}$$

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy = \mathcal{F}^{-1}\{g(y)\}, \tag{6.2}$$

where x is treated as the time variable and y the frequency. We use these symbols to make them universally valid because the Fourier transform can be regarded as *ubiquitous* since it has a wide range of applications in real life problems. The Fourier transform has long been a principle analytical tool in such diverse fields as linear system, probability theory, quantum physics, antennas, distribution theory and signal processing. We can very simply illustrate a practical situation of a rectangular pulse system containing some information in the time domain with the following example.

Example 1

Find the Fourier transform of the rectangular pulse waveform of amplitude A and then discuss its inverse transform analytically and graphically.

Solution

Mathematically we define a rectangular pulse as follows:

$$f(x) = \begin{cases} A, & -x_0 < x < x_0, \\ 0, & |x| \geq x_0. \end{cases}$$

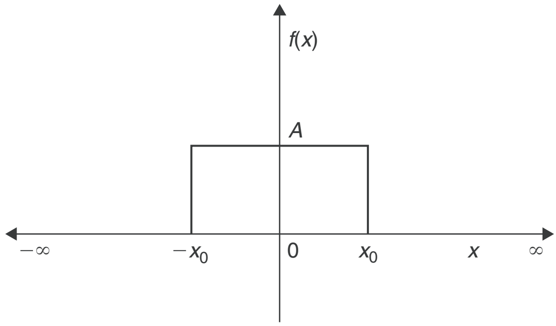

Figure 6.1: A rectangular pulse waveform in the time domain.

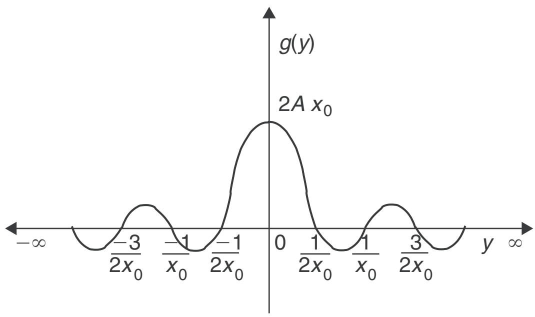

Figure 6.2: Fourier transform of a rectangular pulse waveform in the frequency domain.

If we take the Fourier transform of this pulse we obtain

$$
\begin{aligned}
g(y) &= \int_{-\infty}^{\infty} A \exp(-2\pi ixy)\, dx = \int_{-x_0}^{x_0} A \exp(-2\pi ixy)\, dx \\
&= 2Ax_0 \left\{ \frac{\sin(2\pi x_0 y)}{2\pi x_0 y} \right\} \\
&= 2Ax_0 Sa(2\pi x_0 y).
\end{aligned}
$$

Figure 6.1 depicts the graphical representation of $f(x)$ and Figure 6.2 depicts its Fourier transform $g(y)$. It can be easily seen that this illustration confirms that the Fourier transform frequency domain contains exactly the same information as that of the original function in the time domain; they only differ in the manner of presentation. It is important to note the following condition.

Lemma 1

If $f(x)$ is integrable in the sense

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty, \tag{6.3}$$

then its Fourier transform $g(y)$ exists and satisfies the inverse Fourier transform (6.2).

This example is a classical one which satisfies Lemma 1 stated above and hence we must be able to get the time function $f(x)$ (inverse transform) from the frequency function $g(y)$ (the Fourier transform) using the formula (6.2).

Thus we have

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \left\{ 2Ax_0 \frac{\sin(2\pi x_0 y)}{2\pi x_0 y} \right\} \exp(2\pi ixy)\, dy \\ &= 2Ax_0 \int_{-\infty}^{\infty} \left\{ \frac{\sin(2\pi x_0 y)}{2\pi x_0 y} \right\} [\cos(2\pi xy) + i\sin(2\pi xy)]\, dy. \end{aligned}$$

In this integral the imaginary part is odd and hence its integral will go to zero. We then use the trigonometric identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ and after a little reduction and manipulation we can write

$$\begin{aligned} f(x) = A(x_0 + x) &\int_{-\infty}^{\infty} \frac{\sin(2\pi y(x_0 + x))}{2\pi y(x_0 + x)} dy \\ &+ A(x_0 - x) \int_{-\infty}^{\infty} \frac{\sin(2\pi y(x_0 - x))}{2\pi y(x_0 - x)} dy. \end{aligned}$$

To evaluate this integral we use the well-known formula $\int_{-\infty}^{\infty} \frac{\sin(2\pi bx)}{2\pi bx} dx = \frac{1}{2|b|}$ and we can finally write

$$f(x) = \frac{A}{2}\frac{x_0 + x}{|x_0 + x|} + \frac{A}{2}\frac{x_0 - x}{|x_0 - x|} = f_1(x) + f_2(x),$$

where $f_1(x) = \frac{A}{2}\frac{x_0+x}{|x_0+x|}$ and $f_2(x) = \frac{A}{2}\frac{x_0-x}{|x_0-x|}$.

From Section 2.4 of this book, we can easily rewrite the functions $f_1(x)$ and $f_2(x)$ in terms of signum functions which are related to the Heaviside unit step functions as follows: [Note that these two functions are defined as the generalized functions.]

$$\begin{aligned} f_1(x) &= \frac{A}{2}\frac{x + x_0}{|x + x_0|} = \frac{A}{2} sgn(x + x_0) = \frac{A}{2}\{2H(x + x_0) - 1\}, \\ f_2(x) &= -\frac{A}{2}\frac{x - x_0}{|x - x_0|} = -\frac{A}{2} sgn(x - x_0) = -\frac{A}{2}\{2H(x - x_0) - 1\}. \end{aligned}$$

Hence the final result can be written as

$$\begin{aligned} f(x) &= \frac{A}{2}\{2H(x+x_0) - 1 - [2H(x-x_0) - 1]\} \\ &= A\{H(x+x_0) - H(x-x_0)\}, \end{aligned}$$

which is equivalent to

$$f(x) = \begin{cases} A, & |x| < x_0, \\ 0, & |x| > x_0. \end{cases}$$

Graphical illustrations of the functions $sgn(x \pm x_0)$ and $H(x \pm x_0)$ are displayed in Figure 6.3.

This establishes the Fourier transform pair with the aid of Lemma 1. The reader is referred to the book by Brigham (1974) to consult Lemma 2. We will not pursue any more examples to verify these lemmas.

Lemma 2

If $f(x) = \alpha(x)\sin(2\pi xy + \beta)$ where y and α are arbitrary constants, if $\alpha(x+k) < \alpha(x)$, and if for $|x| > \lambda > 0$, the function $\frac{f(x)}{x}$ is absolutely integrable in the sense of eqn (6.3) then $g(y)$ exists and satisfies the inverse Fourier transform, eqn (6.2).

Example 2

Find the Fourier transform of the following periodic functions and then verify their inverses. (a) $f(x) = A\cos(2\pi y_0 x)$; (b) $f(x) = A\sin(2\pi y_0 x)$.

Solution

Let us first find the Fourier transform of the function $\exp(2\pi i y_0 x)$,

$$\begin{aligned} \mathcal{F}\{\exp(2\pi i y_0 x)\} &= \int_{-\infty}^{\infty} (e^{2\pi i y_0 x}) e^{-2\pi i x y}\, dx \\ &= \int_{-\infty}^{\infty} e^{-2\pi i (y - y_0) x}\, dx \\ &= \left. \frac{e^{-2\pi i (y-y_0)x}}{-2\pi i (y - y_0)} \right|_{-\infty}^{\infty} \\ &= \lim_{x \to \infty} \frac{\sin(2\pi (y - y_0) x)}{\pi (y - y_0)} \\ &= \delta(y - y_0). \end{aligned}$$

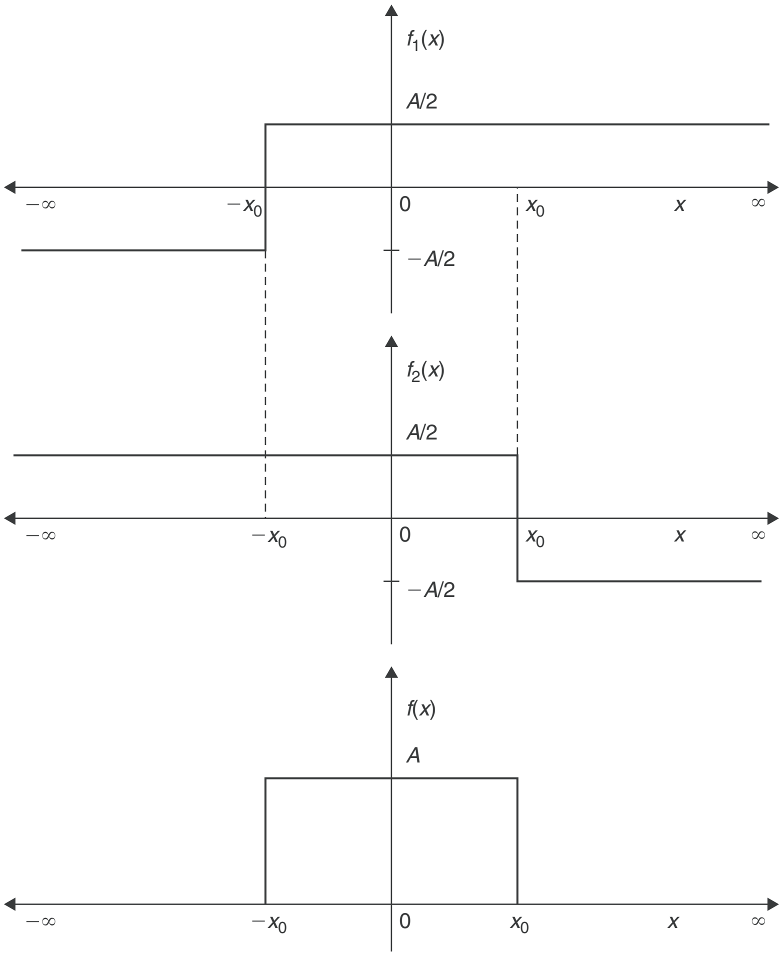

Figure 6.3: Construction of a rectangular pulse from two signum functions. Here $f_1(x) = \frac{A}{2} sgn(x + x_0)$, $f_2(x) = -\frac{A}{2} sgn(x - x_0)$ and $f(x) = A[H(x + x_0) - H(x - x_0)]$.

Similarly,

$$\mathcal{F}\{\exp(-2\pi i y_0 x)\} = \delta(y + y_0).$$

Then using this information, we find that (a)

$$\begin{aligned}\mathcal{F}\{A\cos(2\pi y_0 x)\} &= \frac{A}{2}\mathcal{F}\{e^{2\pi i y_0 x} + e^{-2\pi i y_0 x}\} \\ &= \frac{A}{2}\{\delta(y - y_0) + \delta(y + y_0)\}.\end{aligned}$$

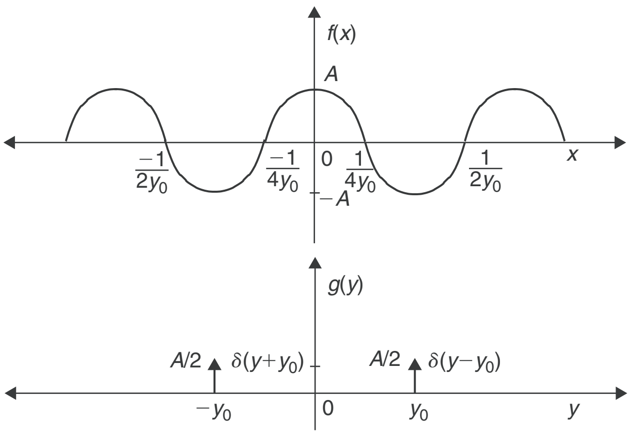

Figure 6.4: A sketch of the Fourier transform of a cosine function. Here $f(x) = A\cos(2\pi y_0 x)$ and $g(y) = \frac{A}{2}[\delta(y-y_0)+\delta(y+y_0)]$.

Similarly, we find that (b)

$$\begin{aligned}\mathcal{F}\{A\sin(2\pi y_0 x)\} &= \frac{A}{2i}\mathcal{F}\{e^{2\pi i y_0 x} - e^{-2\pi i y_0 x}\}\\ &= \frac{A}{2i}\{\delta(y-y_0) - \delta(y+y_0)\}.\end{aligned}$$

The inverse Fourier transforms are then given (a):

$$\begin{aligned}\mathcal{F}^{-1}\left\{\frac{A}{2}[\delta(y-y_0)+\delta(y+y_0)]\right\} &= \frac{A}{2}\left\{\int_{-\infty}^{\infty}[\delta(y-y_0)+\delta(y+y_0)]e^{2\pi i x y}dy\right\}\\ &= \frac{A}{2}[e^{2\pi i x y_0} + e^{-2\pi i y_0 x}]\\ &= A\cos(2\pi y_0 x).\end{aligned}$$

Similarly, we can find the inverse Fourier transform (b):

$$\begin{aligned}\mathcal{F}^{-1}\left\{\frac{A}{2i}[\delta(y-y_0)-\delta(y+y_0)]\right\} &= \frac{A}{2i}\left\{\int_{-\infty}^{\infty}[\delta(y-y_0)-\delta(y+y_0)]e^{2\pi i x y}\,dy\right\}\\ &= \frac{A}{2i}[e^{2\pi i x y_0} - e^{-2\pi i y_0 x}]\\ &= A\sin(2\pi y_0 x).\end{aligned}$$

These results are depicted in Figures 6.4 and 6.5 for the Fourier transform of cosine function and its inverse and the Fourier transform of a sine function and its inverse, respectively.

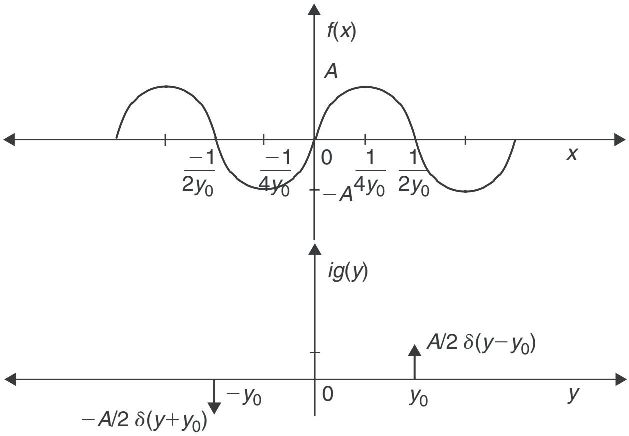

Figure 6.5: A sketch of the Fourier transform of a sine function. Here $f(x) = A\sin(2\pi y_0 x)$ and $g(y) = \frac{A}{2i}[\delta(y - y_0) - \delta(y + y_0)]$.

Lemma 3

It can be stated that all functions for which Lemmas 1 and 2 hold are assumed to be of *bounded variation*; that is, they can be represented by a curve of finite length in any finite time interval. By means of Lemma 3 we extend the theory to include singular (impulse) functions.

If $f(x)$ is a periodic or impulse function, then $g(y)$ exists only if we introduce the theory of distributions. We have already discussed in the previous chapters of this book many generalized functions to deal with the distribution theory. With the aid of those developments the Fourier transform of singular functions can be defined. It is important to develop the Fourier transform of an impulse function because their use greatly simplifies the derivation of many transform pairs.

We know that the impulse function $\delta(x)$ is defined as

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0), \tag{6.4}$$

where $f(x)$ is an arbitrary function continuous at $x = x_0$. Application of definition (6.4) yields Fourier transform of many important functions very easily.

A "unitary function" $U(x)$ can be found, which is a good function vanishing for $|x| \geq 1$ and such that

$$\sum_{n=-\infty}^{\infty} U(x + n) = 1 \tag{6.5}$$

for all x. The Fourier transform $V(y)$ of any such function has $V(0) = 1$, but $V(m) = 0$ if m is an integer other than zero.

Example 3

Prove that the Fourier transform of a sequence of equal distant impulse functions $\delta(x)$ is another sequence of equal distance impulses: Mathematically, if 2ℓ is the period of the sequence of functions, then we need to prove that

$$f(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2\ell n),$$

$$\mathcal{F}\{f(x)\} = g(y) = \frac{1}{2\ell} \sum_{n=-\infty}^{\infty} \delta\left(y - \frac{n}{2\ell}\right).$$

Proof

This sequence of functions is a periodic function with period 2ℓ and so the Fourier series expression can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},$$

$$C_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\, dx,$$

where C_n is the Fourier coefficient.

Now taking the Fourier transform of $f(x)$ we have

$$\mathcal{F}\{f(x)\} = \sum_{n=-\infty}^{\infty} C_n \mathcal{F}\{e^{in\pi x/\ell}\}$$
$$= \sum_{n=-\infty}^{\infty} C_n \delta\left(y - \frac{n}{2\ell}\right).$$

This is the Fourier transform of the sequence of the periodic impulse functions provided we know the value of C_n. Thus to determine this value we follow Lighthill's (1964) approach. Using the concept of *unitary function* introduced by him, we can now evaluate the Fourier coefficient C_n as

$$C_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell) e^{-in\pi x/\ell}\, dx$$
$$= \frac{1}{2\ell} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell) U\left(\frac{x}{2\ell}\right) e^{-in\pi x/\ell}\, dx$$
$$= \frac{1}{2\ell} \sum_{m=-\infty}^{\infty} U(m)$$
$$= \frac{1}{2\ell}.$$

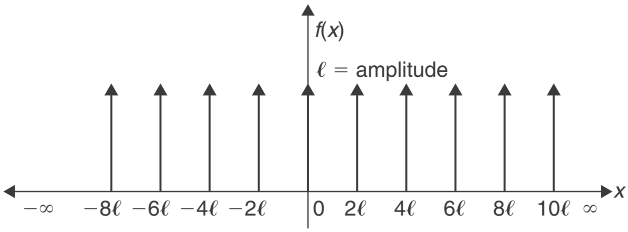

Figure 6.6: A sequence of periodic delta functions $f(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2n\ell)$.

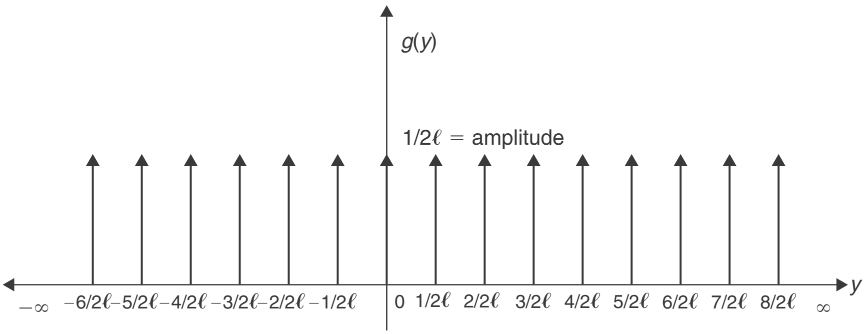

Figure 6.7: A sequence of the Fourier transform of periodic impulse functions $g(y) = \frac{1}{2\ell} \sum_{n=-\infty}^{\infty} \delta(y - \frac{n}{2\ell})$.

Therefore

$$g(y) = \frac{1}{2\ell} \sum_{n=-\infty}^{\infty} \delta\left(y - \frac{n}{2\ell}\right).$$

The Fourier series can be written as

$$f(x) = \frac{1}{2\ell} \sum_{n=-\infty}^{\infty} e^{in\pi x/\ell} = \sum_{n=-\infty}^{\infty} \delta(x - 2n\ell) = \frac{1}{2\ell} + \frac{1}{\ell} \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi x}{\ell}\right).$$

Thus through this investigation we have found an important relationship that

$$2\ell \sum_{n=-\infty}^{\infty} \delta(x - 2n\ell) = 1 + 2 \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi x}{\ell}\right).$$

Figure 6.6 depicts the periodic impulse of the delta function, whereas Figure 6.7 depicts the inverse of this transform as a row of periodic delta functions.

In the following we will demonstrate the Fourier transforms of some elementary functions by the direct integration procedure.

Example 4

Using the direct integration formula of the Fourier transform, determine the Fourier transform of $f(x) = \delta(x)$.

Solution

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} \delta(x)e^{-2\pi ixy}\,dx \\ &= e^{-2\pi iy(0)} = 1.\end{aligned}$$

Thus the Fourier transform of a delta function is 1.

Example 5

Using the direct integration formula of the Fourier transform, determine the Fourier transform of $f(x) = 1$.

Solution

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\ &= \int_{-\infty}^{\infty} (1)e^{-2\pi ixy}\,dx \\ &= \left.\frac{e^{-2\pi ixy}}{-2\pi iy}\right|_{-\infty}^{\infty} \\ &= \lim_{x\to\infty} \frac{e^{2\pi ixy} - e^{-2\pi ixy}}{2\pi iy} \\ &= \lim_{x\to\infty} \frac{\sin(2\pi xy)}{\pi y} \\ &= \delta(y).\end{aligned}$$

Thus the Fourier transform of 1 is a delta function.

Example 6

Using the direct integration formula of the Fourier transform, determine the Fourier transform of $f(x) = H(x)$.

Solution

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\
&= \int_{-\infty}^{\infty} H(x)e^{-2\pi ixy}\,dx \\
&= \int_{0}^{\infty} e^{-2\pi ixy}\,dx \\
&= \frac{e^{-2\pi ixy}}{-2\pi iy}\Big|_0^{\infty} \\
&= \frac{1}{2\pi iy} - \lim_{x\to\infty}\left[\frac{e^{-2\pi ixy}}{2\pi iy}\right] \\
&= \frac{1}{2\pi iy} + \lim_{x\to\infty}\left[\frac{\sin(2\pi xy)}{2\pi y}\right] \\
&= \frac{1}{2}\left[\frac{1}{\pi iy} + \delta(y)\right].
\end{aligned}$$

In evaluating this integral we have used the results

$$\begin{aligned}
\lim_{x\to\infty} \frac{\cos(2\pi xy)}{\pi iy} &= 0, \\
\lim_{x\to\infty} \frac{\sin(2\pi xy)}{\pi y} &= \delta(y).
\end{aligned}$$

Example 7

Using the direct integration formula of the Fourier transform, determine the Fourier transform of a signum function $f(x) = sgn(x)$.

Solution

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx \\
&= \int_{-\infty}^{\infty} sgn(x)e^{-2\pi ixy}\,dx \\
&= \lim_{a\to 0}\int_{-\infty}^{\infty} sgn(x)e^{-a|x|}e^{-2\pi ixy}\,dx \\
&= \lim_{a\to 0}\int_{0}^{\infty} e^{-ax}e^{-2\pi ixy}\,dx - \lim_{a\to 0}\int_{-\infty}^{0} e^{ax}e^{-2\pi ixy}\,dx \\
&= \lim_{a\to 0}\frac{e^{-(a+2\pi iy)x}}{-(a+2\pi iy)}\Big|_0^{\infty} - \lim_{a\to 0}\frac{e^{(a-2\pi iy)x}}{(a-2\pi iy)}\Big|_{-\infty}^{0} \\
&= \frac{1}{i\pi y}.
\end{aligned}$$

In this calculation we have used a smoothing function $e^{-a|x|}$ where $a > 0$.

With the aid of this formula we can very easily determine the Fourier transform $H(x)$ as follows. $H(x) = \frac{1}{2}[1 + sgn(x)]$, and hence

$$\mathcal{F}\{H(x)\} = \frac{1}{2}\mathcal{F}[1 + sgn(x)] = \frac{1}{2}\left[\delta(y) + \frac{1}{i\pi y}\right].$$

Duality theorem

A duality exists between the time domain and the frequency domain. This theorem states that if

$$\mathcal{F}\{f(x)\} = g(y),$$

then

$$\mathcal{F}\{g(x)\} = f(-y).$$

Proof

We know that

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy,$$
$$f(-x) = \int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}\,dy.$$

Now changing the role of x and y, we obtain

$$f(-y) = \int_{-\infty}^{\infty} g(x)e^{-2\pi ixy}\,dx$$
$$= \mathcal{F}\{g(x)\}.$$

Hence the proof.

Example 8

Using duality, prove that if $\mathcal{F}\{sgn(x)\} = \frac{1}{\pi iy}$, then $\mathcal{F}\left\{\frac{1}{x}\right\} = -\pi i\, sgn(y)$.

Proof

By the duality, we can write $\mathcal{F}\{sgn(x)\} = \frac{1}{\pi iy}$, such that $\mathcal{F}\{\frac{1}{i\pi x}\} = sgn(-y) = -sgn(y)$. And hence rearranging the terms, we have

$$\mathcal{F}\left\{\frac{1}{x}\right\} = -i\pi\, sgn(y).$$

This is the required proof.

Example 9

Prove the following Fourier transform pairs:

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(\lambda)\,d\lambda\right\} = \frac{g(y)}{2\pi iy} + \frac{1}{2}g(0)\delta(y), \quad \text{where } g(0) = \int_{-\infty}^{\infty} f(x)\,dx.$$

Proof

We know that

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy.$$

Thus integrating the inverse Fourier transform from $-\infty$ to x, we have

$$\begin{aligned}
\int_{-\infty}^{x} f(\lambda)\,d\lambda &= \int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} g(y)e^{2\pi i\lambda y}\,dy\right\}d\lambda \\
&= \int_{-\infty}^{\infty} g(y)\left\{\int_{-\infty}^{x} e^{2\pi i\lambda y}\,d\lambda\right\}dy \\
&= \int_{-\infty}^{\infty}\left\{\frac{e^{2\pi ixy}}{2\pi iy} + \lim_{\lambda\to\infty}\frac{\sin(2\pi\lambda y)}{2\pi y}\right\}dy \\
&= \int_{-\infty}^{\infty} g(y)\left\{\frac{e^{2\pi ixy}}{2\pi iy} + \frac{1}{2}\delta(y)\right\}dy \\
&= \int_{-\infty}^{\infty}\left\{\frac{g(y)}{2\pi iy}\right\}e^{2\pi ixy}\,dy + \frac{1}{2}\int_{-\infty}^{\infty} g(y)\delta(y)dy \\
&= \mathcal{F}^{-1}\left\{\frac{g(y)}{2\pi iy}\right\} + \frac{1}{2}g(0).
\end{aligned}$$

Now taking the Fourier transform of both sides, we obtain

$$\begin{aligned}
\mathcal{F}\left\{\int_{-\infty}^{x} f(\lambda)\,d\lambda\right\} &= \mathcal{F}\mathcal{F}^{-1}\left\{\frac{g(y)}{2\pi iy}\right\} + \frac{1}{2}g(0)\mathcal{F}(1) \\
&= \frac{g(y)}{2\pi iy} + \frac{1}{2}g(0)\delta(y).
\end{aligned}$$

In evaluating this problem we have used the following well-known formulae:

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{\sin(2\pi\lambda y)}{\pi y} &= \delta(y), \\
\lim_{\lambda\to\infty}\frac{\cos(2\pi\lambda y)}{\pi y} &= 0.
\end{aligned}$$

This is the required proof.

Example 10

Using the direct integration formula of the Fourier transform, determine the Fourier transform of the function $f(x) = e^{-a|x|}$, where $a > 0$.

Solution

Using the direct formula we have

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-2\pi ixy}\, dx \\
&= \int_{-\infty}^{0} e^{(a-2\pi iy)x} + \int_{0}^{\infty} e^{-(a+2\pi iy)x}\, dx \\
&= \frac{1}{a-2\pi iy} + \frac{1}{a+2\pi iy} \\
&= \frac{2a}{a^2+4\pi^2y^2} \\
&= g(y).
\end{aligned}$$

The graphical representation of the functions $f(x)$ and $g(y)$ is depicted in Figure 6.9.

We have so far dealt with the Fourier transform of the continuous functions. In the next section, first we will define with a diagram the discrete function and then develop the Fourier transform formula for this discrete function.

It is worth noting that this example (Example 10) is a very classical one and it has a Fourier transform by analytical procedure. For those problems which do not yield a closed-form Fourier transform solution, we need to take recourse to numerical procedure, that is the DFT. This is done in the following manner.

6.3 The discrete Fourier transform

We develop the discrete function as follows. When $f(x)$, a continuous function, is sampled at a regular interval of period 2ℓ the usual Fourier transform technique is modified. A diagrammatic form of a simple sample together with its associated input–output waveforms is depicted in Figure 6.8.

Let us consider the sampling frequency to be $y_s = \frac{1}{2\ell}$ which is the cycle per second and is called Hertz. Here $f(x)$ is the continuous function and $\hat{f}(x)$ is the discrete version of $f(x)$. Defining the set of impulse functions $\hat{\delta}(x)$ by

$$\hat{\delta}(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2n\ell),$$

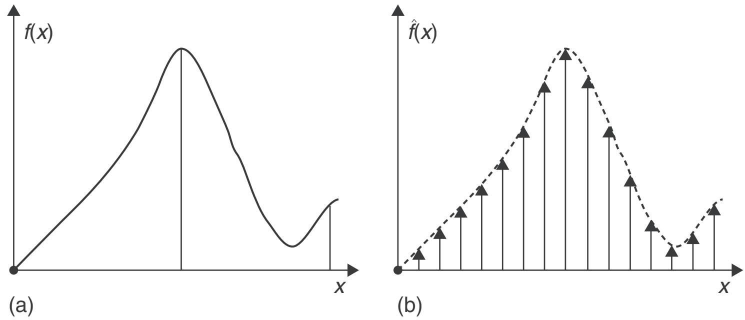

Figure 6.8: (a) A sketch of the continuous function $f(x)$; (b) a sketch of the discrete function $\hat{f}(x)$.

the input–output relationship of the sample becomes

$$\begin{aligned}\hat{f}(x) &= f(x)\hat{\delta}(x) \\ &= f(x)\sum_{n=-\infty}^{\infty}\delta(x-2n\ell) \\ &= \sum_{n=-\infty}^{\infty} f(x)\delta(x-2n\ell) \\ &= \sum_{n=-\infty}^{\infty} f(2n\ell)\delta(x-2n\ell),\end{aligned}$$

where for a given $f(x)$ and (2ℓ), $\hat{f}(x)$ is unique, but the converse is not true. The above infinite series can be truncated to a finite number of terms if we use the following rectangular function of amplitude unity:

$$u(x) = \begin{cases} 1, & -\ell < x < \lambda - \ell, \\ 0, & \text{otherwise}, \end{cases}$$

where λ is the total length of time such that there are N samples each of length 2ℓ and hence $2\ell = \frac{\lambda}{N}$. Thus the above series can be rewritten as follows:

$$\hat{f}(x)u(x) = \sum_{n=0}^{N-1} f(2n\ell)\delta(x-2n\ell),$$

where it has been assumed that there are N equidistant impulse functions lying within the truncation interval, that is, $N = \frac{\lambda}{2\ell}$. The sampled truncation waveform and its Fourier transform are suitable because truncation in the time domain results in *rippling* in the frequency domain.

To get our original transform pair to a DFT pair, we need to modify to sample the Fourier transform of the above equation. In the time domain this product is equivalent to convolving the sample truncated waveform of the above and the time function

$$\hat{\delta}_1(x) = \lambda \sum_{r=-\infty}^{\infty} \delta(x - r\lambda) = \sum_{r=-\infty}^{\infty} e^{2\pi irx/\lambda},$$

because $\hat{\delta}_1(x)$ function is periodic with period λ.

Referring to Brigham (1974), the desired relationship can be written as follows:

$$\begin{aligned}\tilde{f}(x) &= f(x)\hat{\delta}(x)u(x) * \hat{\delta}_1(x) \\ &= \lambda \sum_{r=-\infty}^{\infty} \left\{ \sum_{n=0}^{N-1} f(2n\ell)\delta(x - 2n\ell - r\lambda) \right\}, \end{aligned} \tag{6.6}$$

where $\tilde{f}(x)$ is the approximation to the function $f(x)$. This function is a periodic function with period λ and hence we can expand it as a Fourier series expansion, and this series is given as follows:

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi inx/\lambda}, \tag{6.7}$$

where C_n is the Fourier coefficient and actually it is defined as the DFT. Now the problem reduces to determine this Fourier coefficient, with which our goal will be achieved. Using the definition to find the Fourier coefficient C_n yields

$$\begin{aligned} C_n &= \frac{1}{\lambda} \int_{-\ell}^{\lambda-\ell} \tilde{f}(x) e^{-2\pi inx/\lambda}\, dx \\ &= \frac{1}{\lambda} \int_{-\ell}^{\lambda-\ell} \lambda \sum_{r=-\infty}^{\infty} \left\{ \sum_{k=0}^{N-1} f(2k\ell)\delta(x - 2k\ell - r\lambda) \right\} e^{-2\pi inx/\lambda}\, dx \\ &= \int_{-\ell}^{\lambda-\ell} \left\{ \sum_{k=0}^{N-1} f(2k\ell)\delta(x - 2k\ell) \right\} e^{-2\pi inx/\lambda}\, dx \\ &= \sum_{k=0}^{N-1} f(2\ell k) e^{-2\pi ink/N}, \quad n = 0, \pm 1, \pm 2, \ldots . \end{aligned} \tag{6.8}$$

In evaluating this integral we have used the fact that the integration is done only over one period putting $r = 0$, and that $2\ell = \frac{\lambda}{N}$.

Thus we can write the Fourier transform of the approximate periodic function $\tilde{f}(x)$ as

$$\tilde{g}(y) = \sum_{k=0}^{N-1} f(2\ell k)e^{-2\pi ink/N}, \quad n = 0, 1, 2, \ldots, N-1. \tag{6.9}$$

If we define $y = n\,\Delta y$ such that $\Delta y(2\ell) = \frac{1}{N}$, then the DFT will take the following familiar form:

$$\tilde{g}\left(\frac{n}{2\ell N}\right) = \sum_{k=0}^{N-1} f(2\ell k)e^{-2\pi ink/N}, \quad n = 0, 1, 2, \ldots, N-1. \tag{6.10}$$

Equation (6.10) is the desired DFT; the expression relates N samples of time and N samples of frequency by means of the continuous Fourier transform. The DFT is then a special case of the continuous Fourier transform.

A graphical development of the DFT pairs has been depicted in Figures 6.9 and 6.10. Figure 6.9 is sketched using analytical treatment of the continuous Fourier transform, whereas Figure 6.10 is depicted due to the DFT pair just developed

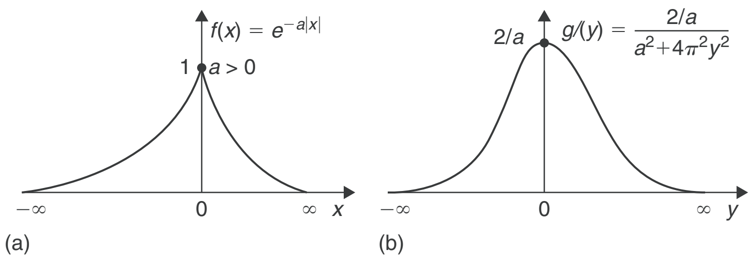

Figure 6.9: (a) The graph of a continuous function $f(x)$; (b) the graph of the Fourier transform of the continuous function $f(x)$, that is, $g(y) = \frac{2a}{a^2 + 4\pi^2 y^2}$.

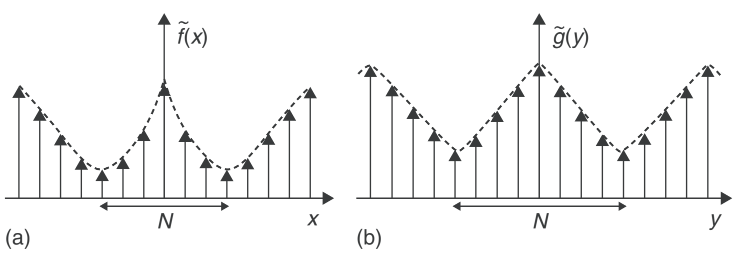

Figure 6.10: Graphical development of the discrete Fourier transform (Brigham, 1974).

(Brigham, 1974). The DFT pair of Figure 6.10 is acceptable for the purpose of the digital machine computation since both the time and frequency domains are repeated by discrete values. In this figure, the original time function, Figure 6.9, is approximated by N samples; the original Fourier transform $g(y)$ is also approximated by N samples. These N samples define the DFT pair and approximate the original Fourier transform pair. It is easily noted from these figures that the sampling of the time and the frequency functions displays that they are periodic functions which should be the case because of the discrete behaviour of the functions. Thus this algorithm seems suitable for the numerical computation. It is to be noted that these forms are predicted by earlier workers in this important field of research.

Now the Fourier transform can be used to define $\hat{f}(x)$ as follows:

$$\begin{aligned}
\hat{g}(y) &= \mathcal{F}\{\hat{f}(x)\} \\
&= \int_{-\infty}^{\infty} \hat{f}(x)e^{-2\pi ixy}\, dx \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(2n\ell)\delta(x-2n\ell)e^{-2\pi ixy}\, dx \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(2n\ell)\delta(x-2n\ell)e^{-2\pi i(2n\ell)y}\, dx \\
&= \sum_{n=-\infty}^{\infty} f(2n\ell)e^{-2\pi i(2n\ell)y}.
\end{aligned}$$

This infinite series is not convenient for machine calculations unless we truncate the series at a certain point with minimum truncation error. Let us consider that the total length of time of experiment is λ and that every (2ℓ) unit of time we take the sample and that there are N samples (N may be a very large number also) which we have recorded, then we have $N = \frac{\lambda}{2\ell}$. Also for the discrete case, we can define $x = n\,\Delta x = (2\ell)n$ and $y = p\,\Delta y$, such that the frequency $\Delta y = \frac{1}{\Delta xN} = \frac{1}{2\ell N}$. Then the above algorithm can be modified using a finite number of terms as follows:

$$\begin{aligned}
\hat{g}(y_p) = \hat{g}\left(\frac{p}{2\ell N}\right) &= \sum_{n=-\infty}^{\infty} f(2n\ell)e^{-2\pi inp/N} \\
&= \sum_{n=0}^{N-1} f(2n\ell)e^{-2\pi inp/N}, \quad p = 0, 1, 2, 3, \ldots, N-1.
\end{aligned}$$

This form will be much more suitable for machine calculation and it is equivalent to eqn (6.10).

We can look at the same problem in another way with mathematical sophistication, as follows.

Since $\hat{\delta}(x)$ is a periodic function, its Fourier series with period 2ℓ can be written as $\hat{\delta}(x) = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} e^{n\pi ix/\ell}$, and hence we have

$$\begin{aligned}
\mathcal{F}\{\hat{f}(x)\} &= \int_{-\infty}^{\infty} \hat{f}(x)e^{-2\pi ixy}\,dx \\
&= \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\ell}\sum_{n=-\infty}^{\infty} e^{n\pi ix/\ell}\right\} e^{-2\pi ixy}\,dx \\
&= \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi ix(y-n/2\ell)}\,dx \\
&= \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} g\left(y-\frac{n}{2\ell}\right) \\
&= \mathcal{F}\{f(x)\} * \mathcal{F}\{\hat{\delta}(x)\}.
\end{aligned}$$

Here the star stands for the convolution integral.

We know that $\mathcal{F}\{\hat{\delta}(x)\} = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}\delta\left(y-\frac{n}{2\ell}\right)$, therefore the above result can be verified as follows:

$$\begin{aligned}
\mathcal{F}\{\hat{f}(x)\} &= [\mathcal{F}\{f(x)\} * \mathcal{F}\{\hat{\delta}(x)\}] \\
&= \int_{-\infty}^{\infty} g(\lambda)\frac{1}{2\ell}\sum_{n=-\infty}^{\infty}\delta\left(y-\frac{n}{2\ell}-\lambda\right)d\lambda \\
&= \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} g\left(y-\frac{n}{2\ell}\right).
\end{aligned}$$

Example 11

Find the DFT of the function $f(x) = e^{-a|x|}$, and depict the graphical representation of the Fourier transform pairs.

Solution

We have already evaluated the Fourier transform of this function in Example 10. Thus using the above result we can at once write the Fourier transform of $\hat{f}(x)$ as follows:

$$\mathcal{F}\{\hat{f}(x)\} = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}\frac{2a}{a^2+4\pi^2\left(y-\frac{n}{2\ell}\right)^2}.$$

6.4 The fast Fourier transform

We have already developed the relationship between the continuous Fourier transform and the DFT in the last section. We illustrate the application of the DFT and then how this transform plays a vital role in the development of the FFT. One does not need any special expertise to formulate the FFT algorithm. The FFT is simply an algorithm, that is, it is just a particular method of performing a series of machine calculations which can compute the DFT much more rapidly than other available algorithms. We will discuss in this section very briefly the computational aspect of the algorithm.

We rewrite eqn (6.10) in a simple form so that a non-expert in this field can very easily understand the algorithm. To simplify the matter, we replace $2\ell k$ by k and $\frac{n}{2\ell N}$ by n so that our new equation will look as follows:

$$g(n) = \sum_{k=0}^{N-1} f(k)e^{-2\pi ink/N}, \quad n = 0, 1, 2, 3, \ldots, N-1. \tag{6.11}$$

This equation implies that it is an $N \times N$ matrix equation, which means it is an algebraic equation of n unknowns. Let us define $e^{-2\pi i/N} = \omega$ so that eqn (6.11) can be rewritten in compact form as

$$\mathbf{g(n)} = \omega^{\mathbf{nk}}\mathbf{f(k)}. \tag{6.12}$$

This is a matrix equation displayed in bold faces. It is to be noted that this equation is a complex equation because ω and $\mathbf{f(k)}$ are complex and so N^2 complex multiplications and $N(N-1)$ complex additions are needed to perform the required matrix computation. The FFT owes its success to the fact that the algorithm reduces the number of multiplications and additions required in the computation of eqn (6.12). Thus we can see that FFT is nothing but the DFT. The only difference is that FFT works much faster than DFT in terms of computer time and efficiency.

For $N = 2^m$ the FFT algorithm is then simply a procedure for factoring an $N \times N$ matrix into m matrices (each $N \times N$) such that each of these factored matrices has the special property of minimizing the number of complex multiplications and additions. If we consider $N = 4 = 2^2$, such that $m = 2$, we note that the FFT requires $N \times m/2 = 4$ complex multiplications and $N \times m = 8$ complex additions, whereas the direct method requires $N^2 = 16$ complex multiplications and $N(N-1) = 4 \times 3 = 12$ complex additions. If we assume that the computing time is proportional to the number of multiplications, the approximate ratio of the direct to FFT computing time is given by

$$\frac{N^2}{N \times m/2} = \frac{2N}{m} = 4.$$

If $N = 1024 = 2^{10} = 2^m$, then $m = 10$ and the ratio $= \frac{2N}{m} = \frac{2\times 1024}{10} = 204.8$, which means the computational reduction of more than 200 to 1. This is a fantastic saving of time by the FFT algorithm.

The FFT algorithm for real data can be found in the book by Brigham (1974) and the interested reader is referred to his work for more insight into the techniques of the FFT program. Concerning the matrix solution of linear algebraic equations, the reader is advised to refer the work of Brebbia (1978).

6.4.1 An observation of the discrete Fourier transform

Another algorithm of the DFT is given below:

$$g(y_k) = \sum_{j=0}^{N-1} f(x_j) \exp(-2\pi i x_j y_k)(x_{j+1} - x_j), \quad k = 0, 1, 2, 3, \ldots, N-1. \quad (6.13)$$

It is a very rough DFT algorithm. This method offers a potential way of getting some approximate solution to the problem if the analytical solution is not available. With this algorithm, however, it has been found after careful investigation that if there are N data points of the function $f(x)$ and if we desire to determine the amplitude of N separate sinusoids, then the computational time is proportional to N^2, the number of multiplications. Even with high speed computers, computations of the DFT require excessive machine time for large N. Thus with the above numerical algorithm, the scientific community worked very hard to reduce the computing time without any success. Then Cooley and Tukey (1965) published their mathematical algorithm, which has been known as the FFT. The FFT is a computational algorithm which reduces the computational time of the DFT given above to a time proportional to $N \log_2 N$. This increase in computing speed has completely revolutionized many facets of scientific analysis. A historical review of the discovery of the FFT illustrates that this important development was almost ignored. As for example if $N = 1024$, then DFT computing time will be $N^2 = 1024^2 = 1\,048\,576$, whereas the FFT computing time will be $N \log_2 N = N \left\{ \frac{\ln N}{\ln 2} \right\} = 1024 \times \frac{\ln 1024}{\ln 2} = 1024 \times 10 = 10\,240$. This is really a fantastic saving in computational time by the FFT algorithm.

6.5 Mathematical aspects of FFT

In this section we shall briefly describe the mathematical development leading to the DFT and then to the FFT. To do this we need to bring the Fourier series pair and the Fourier transform pair which are already developed in the previous chapters for ready reference. The Fourier series pair and the Fourier transform pair are, respectively, given by the following equations.

The Fourier series pair in complex variable form is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},$$
$$C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\,dx. \tag{6.14}$$

The Fourier transforms can be defined in two ways. The first definition is

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx = g(y),$$
$$f(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy = \mathcal{F}^{-1}\{g(y)\}. \tag{6.15}$$

The second definition (conventional) is

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = g(k),$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)e^{ikx}\,dk = \mathcal{F}^{-1}\{g(k)\}. \tag{6.16}$$

The first equation in eqn (6.15) is the Fourier transform and the second equation is its inverse. Similarly, the first in eqn (6.16) is the Fourier transform and the second equation is its inverse. They are equivalent. It is obvious that $2\pi y = k$ such that $dy = \frac{1}{2\pi}dk$.

It is to be noted here that the FFT is a DFT algorithm which reduces the number of computations needed for N points from N^2 to $N\log_2 N$, where $\log_2$ is the base-2 logarithm. FFTs were first discussed by Cooley and Tukey (1965), although Gauss had actually described the critical factorization step as early as 1805.

The DFT version of Fourier transforms can be derived by discretizing eqn (6.15) in the following manner. In the signal processing literature, it is usual practice to write the DFT and its inverse in the more pure form as given below:

$$g(p) = \sum_{n=0}^{N-1} f(n\,\Delta t)\exp(-i2\pi pn\,\Delta t/N), \quad p = 0, 1, 2, \ldots, N-1, \tag{6.17}$$

$$f(n\,\Delta t) = \sum_{p=0}^{N-1} g(p)\exp(i2\pi pn\,\Delta t/N), \quad n = 0, 1, 2, \ldots, N-1, \tag{6.18}$$

where $f(n\,\Delta t)$ denotes the input signal at time (sample) $n\,\Delta t$, and $g(p)$ denotes the pth spectral sample. This form is the simplest mathematically.

6.6 Reviews of some works on FFT algorithms

There are many FFT algorithms that are available in literature. These algorithms are derived from the DFT algorithms and compute exactly in the exact arithmetic, that is, neglecting floating point errors. A few "FFT" algorithms compute the DFT approximately, with an error that can be made arbitrarily small at the expense of increased computations.

An approximate FFT algorithm by Edelman *et al.* (1999) achieves lower computational requirements for parallel computing with the help of a fast multipole method.

Guo & Burrus (1996) takes sparse input/output (time/frequency localization) into account in a wavelet-based approximate FFT more efficiently than is possible with an exact FFT. Another example of an algorithm for approximate computation of a subset of the DFT output is due to Shentov *et al.* (1995). Only the Edelman algorithm works equally well for sparse and non-sparse data. Even the exact FFT algorithms have errors when finite-precision floating-point arithmetic is used, but these errors are typically quite small; most FFT algorithms, for example, Cooley–Tukey, have exactly numerical properties. The upper bound on the relative error for the Cooley–Tukey algorithm is $O(\varepsilon \log N)$, compared to $O(\varepsilon N^{3/2})$ for the naive DFT formula (Gentleman & Sande, 1966) when ε is the machine floating-point relative precision. In fixed-point arithmetic, the finite precision errors accumulated by the FFT algorithms are worse, with root mean square (rms) errors growing as $O(\sqrt{N})$ for the Cooley–Tukey algorithm (Welch, 1969). To verify the correctness of an FFT implementation, rigorous guarantees can be obtained in $O(N \log N)$ time by a simple procedure checking the linearity, impulse-response and time-shift properties of the transform on random input (Ergün, 1995). Next we shall describe here very briefly the algorithms developed by Cooley & Tukey (1965).

6.7 Cooley–Tukey algorithms

So far we have found that the most common FFT is the Cooley–Tukey algorithm. This is a divide and conquer algorithm that recursively breaks down a DFT of any composite size $N = N_1 N_2$ into many smaller DFTs of sizes N_1 and N_2, along with $O(N)$ multiplications by complex roots of unity traditionally called twiddle factors (after Gentleman & Sande, 1966).

This method and the general idea of an FFT was popularized by a publication of Cooley and Tukey in 1965, but it was later discovered (Heideman *et al.*, 1984) that those two authors had independently re-invented an algorithm known to Carl Friedrich Gauss around 1805 (and subsequently rediscovered several times in limited forms).

The most well-known use of the Cooley−Tukey algorithm is to divide the transform into two pieces of size $N/2$ at each step, and is therefore limited to power-of-two sizes, but any factorization can be used in general (as was known to both Gauss and Cooley/Tukey). These are called the *radix-2* and *mixed-radix* cases, respectively. Although the basic idea is recursive, most traditional implementations rearrange the algorithm to avoid explicit recursion. Since the Cooley–Tukey algorithm breaks the DFT into smaller DFTs, it can be combined arbitrarily with any other algorithm for the DFT. We do not want to pursue the other algorithms developed by many other researchers in this book. The interested reader can pursue this matter further by consulting the following link: http://en.wikipedia.org/wiki/Fast_Fourier_transform

6.8 Application of FFT to wave energy spectral density

We shall not go deep into the algorithm of the FFT, rather in the following we will demonstrate an application of wave energy spectral density as described by Chakrabarti (1987).

Random sea state on a short-term basis maintains certain identifiable statistical properties and is best represented by its energy density spectrum. The total energy of a wave E (per unit surface area) in the wave record between infinite time limits is given by the integral

$$E = \frac{1}{2}\rho g \int_{-\infty}^{\infty} |\eta(t)|^2 \, dt, \tag{6.19}$$

where $\eta(t)$ is the wave elevation, ρ is the density and g is the acceleration due to gravity. Chakrabarti (1987) has demonstrated that the energy spectral density by FFT can be obtained as

$$S(\omega) = \frac{1}{T_s} \left| \sum_{n=1}^{N} \eta(n\,\Delta t) \exp(i2\pi f(n\,\Delta t))\,\Delta t \right|^2 . \tag{6.20}$$

Usually in the FFT calculation, the total data length, T_s, is divided into a number of smaller segments, M, each one having an equal number of data points, N, at a constant time increment, Δt. The final result then is averaged over the M sections. The advantage of this method has already been mentioned above. We now discuss what parameters are involved in the computation by FFT. The variables that have to be selected before an energy spectrum of a wave record can be obtained by the FFT algorithm are

- number of sections, M
- number of data points in each section, N (a power of 2)
- time increment or sampling rate, Δt
- frequency increment or resolution, Δf
- frequency range, or so-called Nyquist frequency, f_N.

The first three of these quantities have to be independently selected. The length of the record, T_s, is dependent on M, N and Δt, that is, $T_s = MN\,\Delta t$. For a given record, T_s and Δt are fixed, so that the total number of data points can be obtained

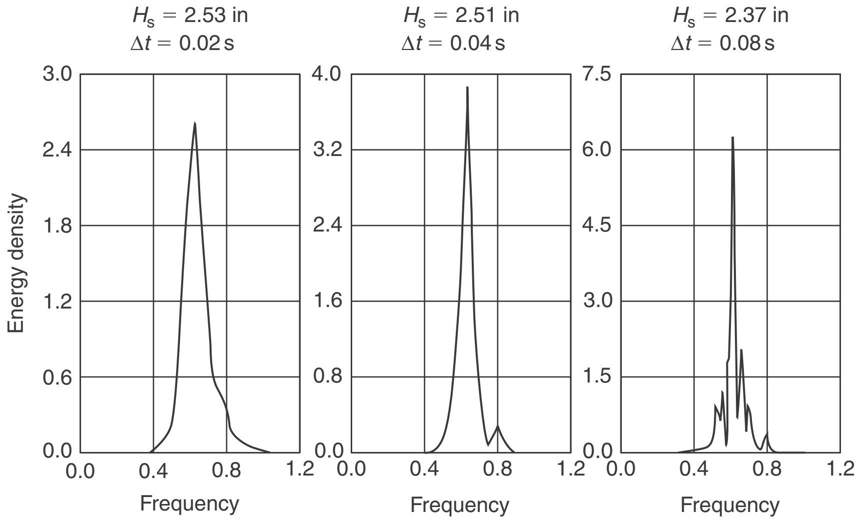

Figure 6.11: Variation of energy density spectral shape of wave record (fixed T_S) with variation of N (or alternatively, Δt for a fixed M). From Chakrabarti (1987).

from these values. Thus, the only choice that has to be made is the number of sections, M. If we know the first three variables, the last two can be calculated as follows:

$$\Delta f = \frac{1}{N(\Delta t)}, \tag{6.21}$$

$$f_N = \frac{1}{2(\Delta t)}. \tag{6.22}$$

Note that the length of wave record is always finite. This requires limiting the Fourier transform in the evaluation of the energy spectrum to a finite Fourier transform. An example of the effect of varying N is shown in Figure 6.11 (Chakrabarti). Note that the energy density spectrum is composed of a finite number of frequencies and higher values of N reveal the individual peaks and reduce the confidence in the ordinary values. The values of H_s (significant wave height), however, is relatively unchanged. In particular, the value of M is usually taken as $M \geq 8$, while the value of N is normally between 512 and 2048.

The book is concerned with the theoretical development of the application of Fourier transforms to generalized functions with a little emphasis on some practical problems. Therefore, we do not want to pursue any rigorous treatment of the numerical algorithm leading to the FFT. The reader is advised to look at the algorithms developed by many scientists and engineers in the last century including the Cooley & Tukey (1965) algorithm available in the literatures.

For further information about this topic, the reader is referred to the following works of Bingham *et al.* (1967), Bracewell (1965), Cooley *et al.* (1967, 1969), Gauss (1866), Gupta (1966), Helms (1967) and Papoulis (1981).

6.9 Exercises

1. By making a substitution of variable show that

$$\int_{-\infty}^{\infty} f(x)\delta(ax - x_0)\, dx = \frac{1}{a} f\left(\frac{x}{a}\right).$$

2. Prove the following Fourier transform pairs:

$$\mathcal{F}\left\{\frac{df(x)}{dx}\right\} = 2\pi i y g(y).$$

3. If $f(x)$ is real, show that $|g(y)|$ is an even function.
4. Find the inverse Fourier transform of the following functions by using the frequency shifting theorem or otherwise:

(a) $$g(y) = \frac{A\sin(2\pi x_0(y - y_0))}{\pi(y - y_0)},$$

(b) $$g(y) = \frac{\beta^2}{\beta^2 + 4\pi^2(y - y_0)^2}.$$

5. Use symmetry theorem and the Fourier transform pairs to determine the Fourier transform of the following:

(a) $$f(x) = \frac{A^2\sin^2(2\pi y_0 x)}{(\pi x)^2},$$

(b) $$f(x) = \frac{\alpha^2}{(\alpha^2 + 4\pi^2 x^2)}.$$

6. Find the Fourier transform of the following functions:

(a) $f(x) = A\cos^2(2\pi y_0 x)$,

(b) $f(x) = A\sin^2(2\pi y_0 x)$.

7. Find the Fourier transform of the following function

$$f(x) = e^{-a|x|}, \quad \text{where } a > 0.$$

8. Prove the following Fourier transform pairs:

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(\lambda)\,d\lambda\right\} = \frac{g(y)}{2\pi iy} + \frac{1}{2}g(0)\delta(y).$$

In evaluating this problem we come across the following well-known formulae. Verify these formulae.

$$\lim_{\lambda\to\infty} \frac{\sin(2\pi\lambda y)}{\pi y} = \delta(y),$$
$$\lim_{\lambda\to\infty} \frac{\cos(2\pi\lambda y)}{\pi y} = 0.$$

9. Show that the Fourier transform of the Gaussian function $f(x) = e^{-ax^2}$ $(a > 0)$ defined for the interval $(-\infty < x < \infty)$ is given by $g(y) = \sqrt{\frac{\pi}{a}}e^{-\pi^2 y^2/a}$ for every y in the interval $(-\infty < y < \infty)$.
10. Let $f(x) = e^{-x}$, $x > 0$. Sample $f(x)$ with $2\ell = 0.01$ and $N = 1024$. Compute the DFT of $g(k)$ with both FFT and DFT. Compare computing times.

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Appendix A: Table of Fourier transforms

A.1 Fourier transforms $g(y) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi ixy}\, dx$

Table A.1: Fourier transforms of some important generalized functions. α stands for any real number not an integer, n for any integer ≥ 0, m for any integer >0 and C for an arbitrary constant.

Formulae	$f(x)$	$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx = g(y)$
1	$\lvert x\rvert^{\alpha}$	$-2\sin\left(\frac{\pi}{2}\alpha\right)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$
2	$\lvert x\rvert^{\alpha}\, sgn(x)$	$-2i\cos\left(\frac{\pi}{2}\alpha\right)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\, sgn(y)$
3	$\lvert x\rvert^{\alpha}H(x)$	$\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\left(e^{-\frac{1}{2}i(\alpha+1)\, sgn(y)}\right)$
4	x^{n}	$(-2\pi i)^{-n}\delta^{(y)}$
5	$x^{n}\, sgn(x)$	$2n!(2\pi iy)^{-n-1}$
6	$x^{n}H(x)$	$(-2\pi i)^{-n}\left\{\frac{1}{2}\delta^{(y)} + \frac{(-1)^{n}n!}{2\pi iy^{n+1}}\right\}$
7	x^{-m}	$-\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\, sgn(y)$
8	$x^{-m}\, sgn(x)$	$-2\frac{(-2\pi iy)^{m-1}}{(m-1)!}(\ln\lvert y\rvert + C)$
9	$x^{-m}H(x)$	$-\frac{(-2\pi iy)^{m-1}}{(m-1)!}\left[\frac{1}{2}\pi i\, sgn(y) + \ln\lvert y\rvert + C\right]$
10	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert$	$\left(2\cos\frac{1}{2}\pi(\alpha+1)\right)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$ $\times\left(-2\ln(2\pi\lvert y\rvert) + \psi(\alpha) - \frac{1}{2}\tan\frac{1}{2}(\alpha+1)\right)$
11	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert\, sgn(x)$	$\left(-2i\sin\frac{1}{2}\pi(\alpha+1)\right)\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}\, sgn(y)$ $\times\left(-2\ln(2\pi\lvert y\rvert) + \psi(\alpha) + \frac{1}{2}\cot\frac{1}{2}(\alpha+1)\right)$
12	$\lvert x\rvert^{\alpha}\ln\lvert x\rvert H(x)$	$(e^{-\frac{1}{2}\pi i(\alpha+1)\, sgn(y)})\alpha!(2\pi\lvert y\rvert)^{-\alpha-1}$ $\times\left\{-\ln(2\pi\lvert y\rvert) + \psi(\alpha) - \frac{1}{2}\pi i\, sgn(y)\right\}$
13	$x^{n}\ln\lvert x\rvert$	$-\pi i\frac{n!}{(2\pi iy)^{n+1}}\, sgn(y)$
14	$x^{n}\ln\lvert x\rvert\, sgn(x)$	$-2\frac{n!}{(2\pi iy)^{n+1}}[\ln(2\pi\lvert y\rvert) - \psi(n)]$
15	$x^{n}\ln\lvert x\rvert H(x)$	$-\frac{n!}{(2\pi iy)^{n+1}}\times\left[\frac{1}{2}\pi i\, sgn(y) + \ln(2\pi\lvert y\rvert) - \psi(n)\right]$
16	$x^{-m}\ln\lvert x\rvert$	$\pi i\frac{(-2\pi iy)^{m-1}}{(m-1)!}\, sgn(y)\times[\ln(2\pi\lvert y\rvert) - \psi(m-1)]$
17	$x^{-m}\ln\lvert x\rvert\, sgn(x)$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!}\times[(\ln(2\pi\lvert y\rvert) - \psi(m-1))^{2} + C]$
18	$x^{-m}\ln\lvert x\rvert H(x)$	$\frac{(-2\pi iy)^{m-1}}{(m-1)!}\left[\frac{1}{2}\left(\frac{1}{2}\pi i\, sgn(y) + \ln(2\pi\lvert y\rvert) - \psi(m-1)\right)^{2} + C\right]$

Table A.2: Fourier transforms of some important elementary generalized functions. $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(x-\lambda)\, d\lambda$ is known as the convolution integral.

Formulae	$f(x)$	$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = g(k)$
19	$\delta(x)$	1
20	$\delta(x-x_0)$	e^{-ikx_0}
21	$e^{\pm ik_0 x}$	$2\pi\delta(k \pm k_0)$
22	1	$2\pi\delta(k)$
23	$\cos k_0 x$	$\pi[\delta(k-k_0)+\delta(k+k_0)]$
24	$\sin k_0 x$	$\frac{\pi}{i}[\delta(k-k_0)-\delta(k+k_0)]$
25	$sgn(x)$	$\frac{2}{ik}$
26	$H(x)$	$\pi\delta(k)+\frac{1}{ik}$
27	$rect(x/x_0)$	$x_0 Sa(kx_0/2) = x_0 \frac{\sin(kx_0/2)}{kx_0/2}$
28	$Sa(x/2)$	$2\pi\, rect(k)$
29	$Sa(x)$	$\pi\, rect(k/2)$
30	$e^{-ax}H(x), a>0$	$\frac{1}{a+ik}$
31	$e^{-a\|x\|}, a>0$	$\frac{2a}{a^2k^2}$
32	$xe^{-ax}H(x), a>0$	$\frac{1}{(a+ik)^2}$
33	$\exp(-x^2/2\omega^2)$	$\omega\sqrt{2\pi}e^{-k^2\omega^2/2}$
34	$\frac{i}{\pi x}$	$sgn(k)$
35	df/dx	$(ik)g(k)$
36	$\int_{-\infty}^{x} f(\lambda)\,d\lambda$	$\frac{1}{ik}g(k)+\pi g(0): \quad g(0)=\int_{-\infty}^{\infty} f(x)\,dx$
37	$f(x \pm x_0)$	$e^{\pm ikx}g(k)$
38	$e^{\pm ik_0 x}f(x)$	$g(k \mp k_0)$
39	$f(\beta x)$	$\frac{1}{\|\beta\|}g\left(\frac{k}{\beta}\right)$
40	$f(x)\cos k_0 x$	$\frac{1}{2}[g(k+k_0)+g(k-k_0)]$
41	$f(x)\sin k_0 x$	$\frac{1}{2i}[g(k-k_0)-g(k+k_0)]$
42	$f_1(x)f_2(x)$	$\frac{1}{2\pi}g_1(k)*g_2(k)$
43	$\int_{-\infty}^{\infty} f_1(\lambda)f_2(x-\lambda)\,d\lambda$	$g_1(k)g_2(k)$
44	$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$	$2\pi\sum_{n=-\infty}^{\infty} c_n\delta\left(k-\frac{n\pi}{\ell}\right)$

Appendix B: Properties of impulse function ($\delta(x)$) at a glance

B.1 Introduction

This appendix contains some important properties of the impulse function ($\delta(x)$). The function is a very important mathematical tool in continuous and discrete Fourier transform analysis as we have already seen in this manuscript. The use of this function simplifies many derivations which would otherwise require lengthy complicated arguments. Even though the concept of the impulse function is correctly applied in the solution of many applied problems, the basics or the definition of impulse is normally mathematically meaningless. Thus to ensure that the impulse function is well defined we must interpret the impulse not as a normal regular function but as a concept in the theory of distributions. We cite the following properties of ($\delta(x)$) function for ready reference.

B.2 Impulse function definition

Usually the impulse function ($\delta(x)$) is defined as

$$\delta(x - x_0) = 0, x \neq x_0, \tag{B.1}$$

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1. \tag{B.2}$$

That is, we define the δ-function as having undefined magnitude at the time of occurrence (here x is treated as time variable) and zero elsewhere with the additional property that area under the function is unity. The concepts are elaborately discussed in Chapters 1 and 2.

B.3 Properties of impulse function

The impulse function $\delta(x - x_0)$ is a distribution assigning to the testing function $f(x)$ the number $f(0)$:

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0). \tag{B.3}$$

It should be reiterated that the relationship (B.3) has no meaning as an integral, but the integral and the function $\delta(x)$ are defined by the number $f(0)$ assigned to the function $f(x)$.

We now describe the useful properties of impulse function.

B.3.1 Sifting property

The function $\delta(x-x_0)$ is defined by

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0). \tag{B.4}$$

This property implies that the δ-function takes on the value of the function $f(x)$ at the time the δ-function is applied. The term sifting property arises in that if we let x_0 continuously vary we can sift out each value of the function $f(x)$. This is the most important property of the δ-function.

B.3.2 Scaling property

The distribution $\delta(ax)$ is defined by

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(x) f\left(\frac{x}{a}\right)dx, \tag{B.5}$$

where the equality results from a change in the independent variable. Thus $\delta(ax)$ is given by

$$\delta(ax) = \frac{1}{|a|}\delta(x). \tag{B.6}$$

B.3.3 Product of a δ-function by an ordinary function

The product of a δ-function by an ordinary function $h(x)$ is defined by

$$\int_{-\infty}^{\infty} [\delta(x)h(x)]\, f(x)\,dx = \int_{-\infty}^{\infty} \delta(x)[h(x)\, f(x)]\,dx. \tag{B.7}$$

If $h(x)$ is continuous at $x=x_0$ then

$$\delta(x_0)h(x) = h(x_0)\delta(x_0). \tag{B.8}$$

In general, product of two distributions is undefined.

B.4 Convolution property

The convolution of two impulse functions is given by

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \delta_1(\lambda)\delta_2(x-\lambda)\, d\lambda \right\} f(x)\, dx = \int_{-\infty}^{\infty} \delta_1(\lambda) \left\{ \int_{-\infty}^{\infty} \delta_2(x-\lambda) f(x)\, dx \right\} d\lambda. \tag{B.9}$$

Hence

$$\delta_1(x-x_1) * \delta_2(x-x_2) = \delta[x-(x_1+x_2)]. \tag{B.10}$$

B.5 δ-Function as generalized limits

Let us consider the sequence $f_n(x)$ of distributions. If there exists a distribution $f(x)$ such that for every test function $h(x)$ we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)h(x)\, dx = \int_{-\infty}^{\infty} f(x)h(x)\, dx, \tag{B.11}$$

then we say that $f(x)$ is the limit of $f_n(x)$:

$$f(x) = \lim_{n\to\infty} f_n(x). \tag{B.12}$$

The δ-function can also be defined as a generalized limit of a sequence of ordinary functions satisfying

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)h(x)\, dx = h(0). \tag{B.13}$$

If eqn (B.13) holds then

$$\delta(x) = \lim_{n\to\infty} f_n(x).$$

Another functional form of importance which defines the δ-function is

$$\delta(x) = \lim_{a\to\infty} \frac{\sin\ ax}{\pi x}. \tag{B.14}$$

Using eqn (B.14) we can prove that

$$\int_{-\infty}^{\infty} \cos(2\pi xy)\, dy = \int_{-\infty}^{\infty} \exp(2\pi ixy)\, dy = \delta(x), \tag{B.15}$$

which is of considerable importance in evaluating particular Fourier transforms.

B.6 Time convolution

If $f(x)$ and $h(x)$ are two functions defined in the limits $-\infty < x < \infty$, then the convolution integral is defined as

$$y(x) = \int_{-\infty}^{\infty} f(\lambda)h(x-\lambda)\,d\lambda = f(x) * h(x).$$

Thus if $y(x)$ is the response obtained by convolving the functions $f(x)$ and $h(x)$, then

$$\mathcal{F}\{y(x)\} = \mathcal{F}\{f(x) * h(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{h(x)\}.$$

B.7 Frequency convolution

A dual to the preceding property is the following:

If $\mathcal{F}\{f_1(x)\} = g_1(y)$ and $\mathcal{F}\{f_2(x)\} = g_2(y)$, then

$$\mathcal{F}\{f_1(x)f_2(x)\} = g_1(y) * g_2(y) = \int_{-\infty}^{\infty} g_1(\lambda)g_2(y-\lambda)\,d\lambda.$$

Thus the inverse transform is given by

$$f_1(x)f_2(x) = \mathcal{F}^{-1}[g_1(y) * g_2(y)] = \int_{-\infty}^{\infty} [g_1(y) * g_2(y)]e^{2\pi ixy}\,dy.$$

Appendix C: Bibliography

This appendix is provided to give the reader some more information about the generalized functions and Fourier transforms including the important area of the fast Fourier transforms. Hope the reader will find it useful.

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Subject index

Integral Equations and their Applications

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For many years, the subject of functional equations has held a prominent place in the attention of mathematicians. In more recent years this attention has been directed to a particular kind of functional equation, an integral equation, wherein the unknown function occurs under the integral sign. The study of this kind of equation is sometimes referred to as the inversion of a definite integral.

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